20. *On Generalized Integrals.* IV

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The $(E.R.)$ integral proposed by Prof. K. Kunugi in [1], that is, the $(E.R.)$ integral in the special sense, which is defined as an extension of the Lebesgue integral, cannot always integrate the important functions: for example, the function $1/x$ is not $(E.R.)$ integrable in the special sense in $[-1, 1]$. Prof. K. Kunugi remarked in [1] that the method of change of the variable admits the extension of the range of the integration. We see the precise definition in [2]. In fact, to do this, he used the function g defined in $[\alpha, \beta]$, which is non-negative and Lebesgue-integrable. Let $G(x)$ be the indefinite integral of g such that $G(\alpha)=a$ and $G(\beta)=b$. For the function $f(t)$ defined in $[a, b]$, if the function $f_1(x)=f(G(x))g(x)$ is $(E.R.)$ integrable in the special sense in $[\alpha, \beta]$, the function $f(t)$ is said to be $(E.R.)$ integrable in the extended sense in $[a, b]$, and we understand by the integral of $f(t)$ in the extended sense in $[a, b]$ the number $(E.R.)\int_\alpha^\beta f(G(x))g(x)dx$. For example, the function $1/t$ is $(E.R.)$ integrable in the extended sense in $[-1, 1]$. For, if we put $g(x)=1/(|x|\log(1/|x|)^2)$, and put $G(x)=1/\log(1/x)$ for $x>0$, $G(0)=0$, $G(x)=-G(-x)$ for $x<0$, then $G(x)$ is the indefinite integral of $g(x)$ such that $G(-1/e)=-1$ and $G(1/e)=1$, and the function $g(x)/G(x)=1/(x\log(1/|x|))$ is $(E.R.)$ integrable in the special sense in $[-1/e, 1/e]$. Hence, the function $1/t$ is $(E.R.)$ integrable in the extended sense in $[-1, 1]$, and the integral is $(E.R.)\int_{-1/e}^{1/e} 1/(x\log(1/|x|))dx=0.$

This theory of Prof. K. Kunugi has been extended to the abstract measure space in [4] by H. Okano. He termed it $(E.R.)$ integral with respect to a measure ν, or $(E.R.\ \nu)$ integral.

In the preceding papers [3], we obtained the set $\boldsymbol{K}$ of special $(E.R.)$ integrable functions as a completion of the set $\mathcal{E}$ of step functions, and showed that the special $(E.R.)$ integral is a continuous linear functional on the complete ranked space $\boldsymbol{K}$. The purpose of this paper is to define the $(E.R.)$ integral in the extended sense in a similar way. Let $\varphi(t)$ be a positive, Lebesgue-integrable function defined in a finite or infinite interval $[a, b]$[1] and $\Phi(t)$ be the indefinite integral of $\varphi(t)$

1) The infinite interval $[a, b]$ designates one of the intervals $-\infty<x<+\infty$, $a\leq x<+\infty$ $(a\neq-\infty)$ and $-\infty<x\leq b$ $(b\neq+\infty)$.

such that $\Phi(\alpha)=a$ and $\Phi(\beta)=b$, and denote by Φ^{-1} the inverse of Φ. Denote by $\mathcal{M}(a, b)$ the set of all measurable functions on $[a, b]$. In order to define the $(E.R.)$ integral in the extended sense, which we will call $(E.R.\ \varphi)$ integral, or $(E.R.)$ integral with respect to φ, first we will introduce the ranked space $\{\mathcal{M}(a, b), \varphi\}$. When $\varphi(t)=1$, $\{\mathcal{M}(a, b), \varphi\}$ coincides with the ranked space $\mathcal{M}$, or $\mathcal{M}(a, b)$, introduced in the consideration of the special $(E.R.)$ integral (see III).[2] In this paper, the set of all $(E.R.\ \varphi)$ integrable functions is given as the r-closure of $\mathcal{E}$ in $\{\mathcal{M}(a, b), \varphi\}$, in other words, the $(E.R.\ \varphi)$ integrable function is defined as the r-limit of a sequence $\{f_n\}$ of step functions in $\{\mathcal{M}(a, b), \varphi\}$. Moreover, the integral is defined as the r-continuous extension of the integrals of step functions. For the consideration of the $(E.R.\ \varphi)$ integral, it is useful to consider the mapping

$$Tf=f(\Phi^{-1}(x))(\Phi^{-1}(x))'$$

from $\mathcal{M}(a, b)$ onto $\mathcal{M}(\alpha, \beta)$. For example, we get the following results: *we have* $f\in\{\lim_n f_n\}$ *in* $\{\mathcal{M}(a, b), \varphi\}$ *if and only if we have* $Tf\in\{\lim_n Tf_n\}$ *in* $\mathcal{M}$. *Moreover, when* $f(x)$ *is* $(E.R.\ \varphi)$ *integrable in* $[a, b]$ *and when* $\{f_n\}$ *is a sequence of step functions converging to* f *in* $\{\mathcal{M}(a, b), \varphi\}$, Tf *is* $(E.R.)$ *integrable in the special sense and the following relation holds*:

$$(E.R.\ \varphi)\int_a^b f(t)dt=\lim_{n\to\infty}\int_a^b f_n(t)dt=\lim_{n\to\infty}\int_\alpha^\beta Tf_n(x)dx=(E.R.)\int_\alpha^\beta Tf(x)dx.$$

7. The extended (E.R.) integrals. First of all, let us consider the set of all measurable functions each of which is defined almost everywhere in a finite or infinite interval $[a, b]$, which we shall denote by $\mathcal{M}$, or, for the purpose of calling special attention to the interval $[a, b]$ in which the functions are defined, by $\mathcal{M}(a, b)$. We also regard two functions equal if they differ only in a set of measure zero. Let $\varphi(t)$ be a positive, Lebesgue-integrable function in $[a, b]$. We introduce on the set $\mathcal{M}$ a set of neighbourhoods in the following way:

Definition 4. Given a closed subset A of $[a, b]$ and a positive number ε, the *neighbourhood* of the point f of $\mathcal{M}$, which we shall denote by $V_\varphi(A, \varepsilon; f)$ (or simply by $V(A, \varepsilon; f)$ if there is no ambiguity about φ), is the set of all those measurable functions $g(t)$ which are expressible as the sums of $f(t)$ and the other functions $r(t)$ with the following properties:

$[\alpha(\varphi)]$ $\quad |r(t)|<\varepsilon\varphi(t)$ for all $t\in A$,

$[\beta(\varphi)]$ $\quad k\int_{\{t;\,|r(t)|>k\varphi(t)\}}\varphi(t)dt<\varepsilon$ for each $k>0$,

$[\gamma(\varphi)]$ $\quad \left|\int_a^b [r(t)]^{k\varphi(t)}dt\right|<\varepsilon$ for each $k>0$,

2) The reference number indicates the number of the Note.

where the function $[r(t)]^{k\varphi(t)}$, the truncation of $r(t)$ by the positive function $k\varphi(t)$, is defined by

$$[r(t)]^{k\varphi(t)}=\begin{cases} r(t) & \text{if } |r(t)|\le k\varphi(t), \\ k\varphi(t)\operatorname{sign} r(t) & \text{if } |r(t)|>k\varphi(t). \end{cases}$$

Then, the neighbourhoods satisfy the axioms (A) and (B) of Hausdorff. We consider the set $\mathcal{M}$ endowed with such a set of neighbourhoods, which we shall denote by $\{\mathcal{M}, \varphi\}$ or $\{\mathcal{M}(a, b), \varphi\}$. In the case of $\varphi(t)=1$, $\{\mathcal{M}, \varphi\}$ coincides with the ranked space $\mathcal{M}$ introduced in the consideration of the special $(E.R.)$ integral (see III). When $\varphi(t)=1$, as in III, we simply denote the space $\{\mathcal{M}, \varphi\}$ by $\mathcal{M}$ or $\mathcal{M}(a, b)$, and simply denote the neighbourhood of f by $V(A, \varepsilon; f)$. Let $x=\Phi(t)$, $t\in[a, b]$, be the indefinite integral of $\varphi(t)$ such that $\Phi(a)=\alpha$ and $\Phi(b)=\beta$,[3] $\alpha, \beta\neq\pm\infty$. Let us consider the following mapping:

$$Tf(x)=f(\Phi^{-1}(x))(\Phi^{-1}(x))' \qquad (x\in[\alpha, \beta]),$$

where Φ^{-1} denotes the inverse of Φ. Then:

Lemma 16. *Tf is a one to one linear mapping of $\mathcal{M}(a, b)$ onto $\mathcal{M}(\alpha, \beta)$, and $\lim_{n\to\infty} f_n(t)=f(t)$ if and only if $\lim_{n\to\infty} Tf_n(x)=Tf(x)$.*

The inverse of T is given in the form: $T^{-1}f=f(\Phi(t))\varphi(t)$. Let us denote, by $\mathcal{L}$ or $\mathcal{L}(a, b)$, the set of all Lebesgue-integrable functions defined on $[a, b]$. Then, the theorem concerning the change of variable in the Lebesgue integral asserts that:

Lemma 17. *$T(\mathcal{L}(a, b))$[4]$=\mathcal{L}(\alpha, \beta)$ and for $f\in\mathcal{L}(a, b)$, $\int_a^b f(t)dt=\int_\alpha^\beta Tf(x)dx$.*

Lemma 18. *For the neighbourhood $V_\varphi(A, \varepsilon; f)$ in $\{\mathcal{M}, \varphi\}$, $T(V_\varphi(A, \varepsilon; f))$ is a neighbourhood in $\mathcal{M}$, and we have*

$$T(V_\varphi(A, \varepsilon; f))=V(\Phi(A), \varepsilon; Tf).$$

Similarly, we have

$$T^{-1}(V(A, \varepsilon; f))=V_\varphi(\Phi^{-1}(A), \varepsilon; T^{-1}f).$$

Proof. In general, we have, for the measurable function $r(t)$, $t\in[a, b]$, the following three properties: (i) $|r(t)|<\varepsilon\varphi(t)$ for all $t\in A$ if and only if $|Tr(x)|<\varepsilon$ for all $x\in\Phi(A)$. (ii) We have $\int_{\{t;\,|r(t)|>k\varphi(t)\}}\varphi(t)dt=\text{mes}\{x; |Tr(x)|>k\}$ for $k>0$. This follows from the theorem concerning the change of variable in the Lebesgue integral. Similarly, we have (iii) $\left|\int_a^b [r(t)]^{k\varphi(t)}dt\right|=\left|\int_\alpha^\beta [Tr(x)]^k dx\right|$ for $k>0$. Thus, from (i), (ii), (iii) and Lemma 16, the desired assertion follows.

3) We see that the definition of $(E.R.\ \varphi)$ integral does not depend on the particular choice of the indefinite integral $\Phi(t)$.

4) In general, when $\pi(p)$ is a mapping defined on R, for a subset E of R, $\pi(E)$ denotes the set $\{\pi(p); p\in E\}$.

By this Lemma, we see that $\{\mathcal{M}, \varphi\}$ is a space of depth ω_0, since $\mathcal{M}$ is a space of depth ω_0. Hence, the indicator should be ω_0. For $n=0, 1, 2, \cdots$, a neighbourhood $V_\varphi(A, \varepsilon; f)$ is said to be rank n, if it satisfies the following condition

$$[\delta(\varphi)] \qquad \int_{[a,b]\setminus A} \varphi(t)dt < \varepsilon \quad \text{and} \quad \varepsilon = 2^{-n}.$$

We have the relation $\int_{[a,b]\setminus A} \varphi(t)dt = \text{mes}([\alpha, \beta]\setminus \Phi(A))$, therefore it follows that:

Lemma 19. *$V_\varphi(A, \varepsilon; f)$ is a neighbourhood of f of rank n in $\{\mathcal{M}, \varphi\}$ if and only if $T(V_\varphi(A, \varepsilon; f))$ is a neighbourhood of Tf of rank n in $\mathcal{M}$.*

By III, Proposition 4 and Lemmas 18, 19, we have:

Proposition 12. *$\{\mathcal{M}, \varphi\}$ is a ranked space of depth ω_0.*

Lemma 20. *$\{V_\varphi(A_n, \varepsilon_n; f_n)\}$ is fundamental in $\{\mathcal{M}, \varphi\}$ if and only if $\{T(V_\varphi(A_n, \varepsilon_n; f_n))\}$ is fundamental in $\mathcal{M}$.*

Thus, we see that:

Proposition 13. *The mapping T is an r-isomorphism of $\{\mathcal{M}, \varphi\}$ onto $\mathcal{M}$.*

From this, paying attention to Lemma 16 and III, Lemmas 12, 13, we infer the following two Lemmas:

Lemma 21. *If $\{f_n\}$ is an r-converging sequence in $\{\mathcal{M}, \varphi\}$, then the limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists and $\{\lim_n f_n\}$ is the set consisting of f alone.*

Lemma 22. *If $f \in \{\lim_n f_n\}$ and $g \in \{\lim_n g_n\}$ in $\{\mathcal{M}, \varphi\}$, then we have $\alpha f + \beta g \in \{\lim_n (\alpha f_n + \beta g_n)\}$ in $\{\mathcal{M}, \varphi\}$ for any pair, α and β, of real numbers.*

As in the preceding papers, denote by $\mathcal{E}$ or $\mathcal{E}(a, b)$, the set of all step functions on $[a, b]$.[5] We now consider the set of such functions which are defined as r-limits of the sequences $\{f_n\}$ of points of $\mathcal{E}$ in $\{\mathcal{M}, \varphi\}$, that is, the r-closure of $\mathcal{E}$ in $\{\mathcal{M}, \varphi\}$. We denote the set by $\boldsymbol{K}(\varphi)$, and call the function belonging to $\boldsymbol{K}(\varphi)$ *(E.R.) integrable function with respect to φ*, or *(E.R. φ) integrable function*. When $\varphi(t)=1$, we simply denote, by $\boldsymbol{K}$, the set $\boldsymbol{K}(\varphi)$, that is, the set of all special $(E.R.)$ integrable functions, as in the preceding papers. Then, we have:

Lemam 23. $Cl_r(\boldsymbol{K}(\varphi)) = \boldsymbol{K}(\varphi)$.

Proof. Let $f \in Cl_r(\boldsymbol{K}(\varphi))$. Then, there is a sequence $\{f_n\}$ of

5) In the case of an infinite interval $[a, b]$, for example, $-\infty > x > +\infty$, step functions are functions having a constant value α_i in each of finite number of sub-intervals (finite or infinite) $a_{i-1} < x < a_i$ in a division of $-\infty < x < +\infty$: $a_0 = -\infty < a_1 < \cdots < a_n = +\infty$.

points of $\boldsymbol{K}(\varphi)$ converging to f, and so there is a fundamental sequence $\{V_\varphi(A_n, \varepsilon_n; f)\}$ in $\{\mathcal{M}, \varphi\}$ such that $V_\varphi(A_n, \varepsilon_n; f) \ni f_n$ and $\varepsilon_n \geq 2\varepsilon_{n+1}$. Since $f_n \in \boldsymbol{K}(\varphi)$, there are a point f_n^* of $\mathcal{E}$ and a neighbourhood $V_\varphi(A_n^*, \varepsilon_n^*; f_n)$ of f_n in $\{\mathcal{M}, \varphi\}$ which has f_n^* as a member and satisfies the relations: $\varepsilon_n^* \leq \varepsilon_n$ and $\int_{[a,b]\setminus A_n^*} \varphi(t)dt < \varepsilon_n^*$. Then $\{V_\varphi(A_n \cap (\bigcap_{m=n}^{\infty} A_m^*),$ $2^3\varepsilon_n; f)\}$ is fundamental in $\{\mathcal{M}, \varphi\}$ and we have $V_\varphi(A_n \cap (\bigcap_{m=n}^{\infty} A_m^*), 2^3\varepsilon_n; f)$ $\ni f_n^*$, which shows that $f \in \boldsymbol{K}(\varphi)$.

In particular, we can see, from this Lemma, that $Cl_r(\mathcal{L}) = \boldsymbol{K}(\varphi)$ in $\{\mathcal{M}, \varphi\}$, because, as it is easily seen, if f is Lebesgue-integrable, it is also $(E.R.\ \varphi)$ integrable. Therefore, from Lemma 17 and Proposition 13, we have the following proposition:

Proposition 14. $T(\boldsymbol{K}(\varphi)) = \boldsymbol{K}$.

This result asserts that in order that f should be $(E.R.\ \varphi)$ integrable in $[a, b]$, it is necessary and sufficient that Tf should be $(E.R.)$ integrable in $[\alpha, \beta]$. By Lemma 16, Proposition 14 and III, Proposition 6, we have:

Proposition 15. *$\boldsymbol{K}(\varphi)$ is a vector space.*

Now, we see, for every $(E.R.\ \varphi)$ integrable function f, that if $\{f_n\}$ is a sequence of points of $\mathcal{E}$ r-converging to f in $\{\mathcal{M}, \varphi\}$, then the sequence of integrals $\int_a^b f_n(x)dx$ converges to a finite limit. In fact, since $f \in \{\lim_n f_n\}$ in $\{\mathcal{M}, \varphi\}$, we have, by Proposition 13, $Tf \in \{\lim_n Tf_n\}$ in $\mathcal{M}$. Moreover, since $Tf_n \in \mathcal{L}$ and $Cl_r(\mathcal{L}) = \boldsymbol{K}$ in $\mathcal{M}$, Tf is $(E.R.)$ integrable in the special sense and we have $\lim_{n\to\infty} \int_\alpha^\beta Tf_n(x)dx = (E.R.)$ $\int_\alpha^\beta Tf(x)dx$. On the other hand, we have, by Lemma 17, $\int_a^b f_n(t)dt$ $= (E.R.) \int_\alpha^\beta Tf_n(x)dx$. Consequently, $\lim_{n\to\infty} \int_a^b f_n(t)dt$ exists and the value coincides with the value $(E.R.) \int_\alpha^\beta Tf(x)dx$. This indicates that the limit does not depend on the particular choice of the sequence $\{f_n\}$ of points of $\mathcal{E}$ converging to f. We understand by the *integral* of the function f the limit value, and we denote the value by $(E.R.\ \varphi)$ $\int_a^b f(t)dt$.

As an immediate consequence of the definition, we have:

Proposition 16. *If f is $(E.R.\ \varphi)$ integrable, then we have*

$$(E.R.\ \varphi) \int_a^b f(t)dt = (E.R.) \int_\alpha^\beta Tf(x)dx.$$

Example 1. If we put $\varphi(t) = \frac{1}{t^2} e^{-\frac{1}{|t|}}$, $t \in [-1, 1]$, the function $1/t$

is $(E.R.\ \varphi)$ integrable in $[-1, 1]$, and we have $(E.R.\ \varphi)\int_{-1}^{1} 1/t\,dt = (E.R.)\int_{-1/e}^{1/e} 1/(x\log(1/|x|))\,dx = 0.$

Let us introduce on $\boldsymbol{K}(\varphi)$ and $\mathcal{E}$ the sets of neighbourhoods and the ranks induced from $\{\mathcal{M}, \varphi\}$ respectively. We denote these ranked spaces by $\{\boldsymbol{K}(\varphi), \varphi\}$ and $\{\mathcal{E}, \varphi\}$. Then, they are ranked subspaces of $\{\mathcal{M}, \varphi\}$. We also see that $\{\mathcal{E}, \varphi\}$ is a ranked subspace of $\{\boldsymbol{K}(\varphi), \varphi\}$. Therefore, from Lemmas 18, 19, Proposition 14 and III, Lemma 15, we infer that:

Theorem 5. *$\{\boldsymbol{K}(\varphi), \varphi\}$ is a completion of $\{\mathcal{E}, \varphi\}$.*

Moreover, from Lemma 16, Propositions 13, 14, 16 and III, Theorem 4, we obtain that:

Theorem 6. *The $(E.R.\ \varphi)$ integral is an r-continuous linear functional on $\{\boldsymbol{K}(\varphi), \varphi\}$.*

Since we have $Cl_r(\mathcal{L}) = \boldsymbol{K}(\varphi)$, we see, by Theorem 6, that $(E.R.\ \varphi)$ integral is defined as the r-continuous extension of Lebesgue integral.

From Proposition 14 and II, Theorem 2, we infer that:

Theorem 7. *In order that a function $f(t)$ should be $(E.R.\ \varphi)$ integrable in $[a, b]$, it is necessary and sufficient that the function should satisfy the following two conditions:*

$$[A_1(\varphi)]\ \lim_{k\to\infty} k\int_{\{t;|f(t)|>k\varphi(t)\}} \varphi(t)dt = 0,$$

$$[A_2(\varphi)]\ \lim_{k\to\infty} \int_a^b [f(t)]^{k\varphi(t)}\,dt \text{ exists},$$

where k runs through the set of all positive integers.

If ν is the measure defined by $\nu(B) = \int_B \varphi(t)dt$ in $[a, b]$, from H. Yamagata [5] and Theorem 7, we see that:

Corollary 3. *The $(E.R.\ \nu)$ integral coincides with the $(E.R.\ \varphi)$ integral.*

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