37. On Generalized (A)-integrals. I

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1. Introduction. To consider conjugate functions E.C. Tichmarsh introduced, in [1], the (Q)-integral. We say that $f(x)$ is (Q)-integrable in $[a, b]$ when there exists $\lim_{n\to\infty}\int_a^b [f(x)]_n dx$ and it is finite, and the limit is denoted by $(Q)\int_a^b f(x)dx$. But the (Q)-integral does not possess the additive property of integral. A.N. Kolmogorov showed, in [2], that if (Q)-integrable functions $f_i(x)$ $(i=1, 2)$ satisfies the condition: $n\ \mathrm{mes}\,(x\,;|f_i(x)|\geqq n)=o(1)$ $(i=1,2)$, for any α_i $(i=1,2)$, $\sum_i \alpha_i f_i(x)$ is also (Q)-integrable and $(Q)\int_a^b \sum_i \alpha_i f_i(x)dx=\sum_i \alpha_i (Q)\int_a^b f_i(x)dx$. If a (Q)-integrable function $f(x)$ satisfies the above condition, we say that $f(x)$ is $(\mathbf{A})$-integrable in $[a, b]$, and give a value of the $(\mathbf{A})$-integral by that of the (Q)-integral. A Lebesgue integrable function is $(\mathbf{A})$-integrable and both integrals have the same value. But there exists a function which is not $(\mathbf{A})$-integrable, for example $g(x)=(-1)^n/x$ where $1/n+1<x\leqq 1/n$ $(n=1, 2, \cdots)$ and $g(0)=0$. K. Kunugi has proposed in [3] the notion of the generalized $(E.R.)$-integral by which this $g(x)$ is integrable in $[0,1]$.

In this paper, we state a generalization of the $(\mathbf{A})$-integral.

2. The generalization of (A)-integral. In this paper, consider only real valued functions which are measurable and almost everywhere finite in $[0,1]$ and denote the set of these functions by $\mathfrak{M}[0,1]$. Let $\mathfrak{H}\equiv\{h_n(x)\}_{n=1,2\cdots}$ be a sequence of non-negative Lebesgue integrable functions tending to infinite almost everywhere in $[0,1]$.

Definition of the $(\mathbf{A}, \mathfrak{H})$*-integral. We say that* $f(x)$ *of* $\mathfrak{M}[0, 1]$ *is* $(\mathbf{A}, \mathfrak{H})$*-integrable in* $[0, 1]$ *if* $f(x)$ *satisfies following* $[a]$ *and* $[b]$:

$[a]$ $\int_{(x;|f(x)|\geqq \alpha h_n(x))} h_n(x)dx=o(1)$ *for any* $\alpha>0$,

$[b]$ $\lim_{n\to\infty}\int_0^1 [f(x)]_{h_n}dx$ *exists and is finite, where*

$[f(x)]_{h_n}=f(x)$ *for* $|f(x)|<h_n(x)$ *and* $=0$ *for* $|f(x)|\geqq h_n(x)$.

The value of the integral is given by this limit and we denote it by $(\mathbf{A}, \mathfrak{H})\int_0^1 f(x)dx$.

Especially put $h_n(x)=n\ u(x)$, where $u(x)$ is positive and Lebesgue integrable, we can replace [a] by following $[a']$: $[a']\ n\int_{(x;|f(x)|\geqq nu(x))} u(x)dx = o(1)$. The generalization in this form has already been gotten by H. Yamagata [4].

The $(\mathbf{A}, \mathfrak{H})$-integral has a few properties which the $(\mathbf{A})$-integral has.

Proposition 1. *The $(\mathbf{A}, \mathfrak{H})$-integral possess the additive property of integral.*

Proposition 2. *Let $\{f_n(x)\}$ be a sequence of $(\mathbf{A}, \mathfrak{H})$-integrable functions satisfying (1) $f_1(x) \leqq f_2(x) \leqq \cdots$ and (2) $(\mathbf{A}, \mathfrak{H}) \int_0^1 f_n(x)dx$ are uniformly bounded, and put $\lim_{n\to\infty} f_n(x) = f(x)$, then $f(x)$ is $(\mathbf{A}, \mathfrak{H})$-integrable and $(\mathbf{A}, \mathfrak{H}) \int_0^1 f(x)dx = \lim_{n\to\infty} (\mathbf{A}, \mathfrak{H}) \int_0^1 f_n(x)dx$.*

We introduce some notations as follows.

(1) *When there are two integrals, X integral and Y integral, if any X integrable function is Y integrable and both integrals have the same value, we write $(X) \lhd (Y)$, and if the converse is also true, we write $(X) = (Y)$.*

(2) $p(x) \vee q(x) \equiv \max(p(x), q(x))$, $p(x) \wedge q(x) \equiv \min(p(x), q(x))$. *When* $\mathfrak{H}^{(i)} \equiv \{h_n^{(i)}(x)\}_{n=1,2,\dots}$ $(i=1, 2)$,

(3) $\mathfrak{H}^{(1)} \vee \mathfrak{H}^{(2)} \equiv \{h_n^{(1)}(x) \vee h_n^{(2)}(x)\}$, $\mathfrak{H}^{(1)} \wedge \mathfrak{H}^{(2)} \equiv \{h_n^{(1)}(x) \wedge h_n^{(2)}\}$ *and* $c_1\mathfrak{H}^{(1)} + c_2\mathfrak{H}^{(2)} \equiv \{c_1 h_n^{(1)} + c_2 h_n^{(2)}\}$ *where* $c_i > 0$ $(i=1, 2)$.

Proposition 3. *Denote the Lebesgue integral by L, we have $(L) \lhd (\mathbf{A}, \mathfrak{H})$ for any $\mathfrak{H}$.*

Proposition 4. *A non-negative $(\mathbf{A}, \mathfrak{H})$-integrable function is Lebesgue integrable.*

Proposition 5. *If $\mathfrak{H}_s$ is a sub-sequence of $\mathfrak{H}$, we have $(\mathbf{A}, \mathfrak{H}) \lhd (\mathbf{A}, \mathfrak{H}_s)$.*

Proposition 6. *For any positive number c, $(\mathbf{A}, \mathfrak{H}) = (\mathbf{A}, c\mathfrak{H})$.*

Proposition 7. *When $f(x)$ is $(\mathbf{A}, \mathfrak{H}^{(i)})$-integrable $(i=1,2)$, if it is $(\mathbf{A}, \mathfrak{H}^{(1)} + \mathfrak{H}^{(2)})$-ntegrable, it is $(\mathbf{A}, \mathfrak{H}^{(1)} \vee \mathfrak{H}^{(2)})$-integrable. The converse is true. And $(A, \mathfrak{H}^{(1)} + \mathfrak{H}^{(2)}) \int_0^1 f(x)dx = (\mathbf{A}, \mathfrak{H}^{(1)} \vee \mathfrak{H}^{(2)}) \int_0^1 f(x)dx$.*

When $h^{(1)}(x)$ and $h^{(2)}(x)$ are both non-negative and Lebesgue integrable, we have following formulae,

(1) $$\int_{(x;|f(x)|\geqq h^{(1)}(x)\vee h^{(2)}(x))} h^{(1)}(x) \vee h^{(2)}(x)dx \leqq \sum_i \int_{(x;|f(x)|\geqq h^{(2)}(x))} h^{(i)}(x)dx$$
$$\int_{(x;|f(x)|\geqq h^{(i)}(x)\wedge h^{(2)}(x))} h^{(1)}(x) \wedge h^{(2)}(x)dx \leqq \sum_i \int_{(x;|f(x)|\geqq h^{(2)}(x))} h^{(i)}dx,$$

(2) $$\sum_i \int_{(x;|f(x)|\geqq h^{(i)}(x))} h^{(i)}(x)dx \leqq \int_{(x;|f(x)|\geqq h^{(1)}(x)\vee h^{(2)}(x))} h^{(1)}(x) \vee h^{(2)}(x)dx$$
$$+ \int_{(x;|f(x)|\geqq h^{(1)}(x)\wedge h^{(2)}(x))} h^{(1)}(x) \wedge h^{(2)}(x)dx,$$

(3) $\sum_i \int_0^1 [f(x)]_{h^{(i)}} dx = \int_0^1 [f(x)]_{h^{(1)} \vee h^{(2)}} dx$
$+ \int_0^1 [f(x)]_{h^{(1)} \wedge h^{(2)}} dx.$

Then, we have the following proposition 8.

Proposition 8. (1) *If a function of* $\mathfrak{M}[0,1]$ *satisfies the condition* [a] *with respect to* $\mathfrak{H}^{(i)}$ $(i=1,2)$, *it also satisfies* [a] *with respect to* $\mathfrak{H}^{(1)} \vee \mathfrak{H}^{(2)}$ *and* $\mathfrak{H}^{(1)} \wedge \mathfrak{H}^{(2)}$. *And the converse is true.* (2) *If* $f(x)$ *is integrable in the three senses of* $(\mathbf{A}, \mathfrak{H}^{(1)})$-, $(\mathbf{A}, \mathfrak{H}^{(2)})$, $(\mathbf{A}, \mathfrak{H}^{(1)} \vee \mathfrak{H}^{(2)})$- *and* $(\mathbf{A}, \mathfrak{H}^{(1)} \wedge \mathfrak{H}^{(2)})$-*integral, it is integrable in the other sense, and* $\sum_i (\mathbf{A}, \mathfrak{H}^{(i)}) \int_0^1 f(x)dx = (\mathbf{A}, \mathfrak{H}^{(1)} \vee \mathfrak{H}^{(2)}) \int_0^1 f(x)dx + (\mathbf{A}, \mathfrak{H}^{(1)} \wedge \mathfrak{H}^{(2)}) \int_0^1 f(x)dx.$
In the special case, when $\{I_n\}$ is a sequence of measurable sets satisfying (1) $I_1 \subseteq I_2 \cdots \subseteq [0,1]$ and (2) $\lim_{n\to\infty} \mathrm{mes}\, I_n = 1$, $h_n(x) = n$ in I_n and $=0$ in I_n^c. In this case, we call the $(\mathbf{A}, \mathfrak{H})$-integral $(\mathbf{A}, I_n)$-integral.

Theorem 1. *If* $f(x)$ *is a function of* $\mathfrak{M}[0,1]$, *there exists a sequence* $\{I_n\}$ *of measurable sets which satisfies above two conditions and* (1) $n\, \mathrm{mes}\, \{(x\,;\, |f(x)| \geqq \alpha n) \cap I_n\} = o(1)$ *for any* $\alpha > 0$. (2) $\lim_{n\to\infty} \int_{I_n} [f(x)]_n dx$ *exists. If this limit is finite,* $f(x)$ *is* $(\mathbf{A}, I_n)$-*integrable in* $[0,1]$.

Proof. At first, take a sequence $\{\varepsilon_n\}$ of monotone decreasing positive numbers tending to zero. For ε_1, there is a measurable set I_1^* in which $f(x)$ is bounded, and $\mathrm{mes}\, I_1^* > 1 - \varepsilon_1$. Put $m_1^* = \left[\max_{x \in I_1^*} |f(x)|\right]$[1] $+1$ and $\tilde{I}_1 = I_1^*$. For ε_2, we get I_2^* as same as I_1^*, and put $m_2^* = \max\left\{m_1^*, \left[\max_{x \in I_2^*} |f(x)|\right] + 1\right\}$ and $\tilde{I}_2 = \tilde{I}_1 \cup I_2^*$. In the same manner, construct $\{m_k^*\}$ and $\{I_k\}$, and let $\{\alpha_k\}$ be a sequence of positive numbers monotone increasing to infinite and $\alpha_1 = 1$. For any $\alpha > 0$, $k(\alpha)$ be the smallest s such that $[\alpha_s] > \dfrac{1}{\alpha}$, $|f(x)| < m_k^*$ in I_k when $k > k(\alpha)$. Put $\tilde{m}_k = [\alpha_k m_k^*]$, when $k > k(\alpha)$, $\mathrm{mes}\, \{(x\,;\, |f(x)| \geqq \alpha m_k) \cap I_k\} = 0$ since $\alpha \tilde{m}_k > m_k$. When $\tilde{m}_k \leqq m < \tilde{m}_{k+1}$, put $I'_m = \tilde{I}_k$, $m\, \mathrm{mes}\, \{(x\,;\, |f(x)| \geqq \alpha m) \cap I'_m\} = 0$ for $m > \tilde{m}_{k(\alpha)}$. We easily get a sub-sequence $\{m_p\}$ of integers such that $\lim_{p\to\infty} \int_{I'_{m_p}} [f(x)]_{m_p} dx$ exists. When $m_p \leqq n < m_{p+1}$, put $I_n = I'_{m_p}$, then $\left| \int_{I'_{m_p}} [f(x)]_{m_p} dx - \int_{I_n} [f(x)]_n dx \right| \leqq n\, \mathrm{mes}\, \{(x\,;\, |f(x)| \geqq m_p) \cap I'_{m_p}\} = 0$ for $m_p > \tilde{m}_{k(1)}$. It is obvious that $n\, \mathrm{mes}\, \{(x\,;\, |f(x)| \geqq \alpha n) \cap I_n\} = o(1)$ for $\alpha > 0$. (Q.E.D.)

Corollary. *If* $f(x)$ *and* $g(x)$ *are in* $\mathfrak{M}[0,1]$, *there is a sequence*

1) $[x]$ is the integral part of x.

$\{I_n\}$ of measurable sets fulfiling the theorem with respect to $f(x)$ and $g(x)$.

If $f(x)$ is in $\mathfrak{M}[0,1]$ and not Lebesgue integrable, we have a problem, for given c, whether we can construct an $(\mathbf{A}, I_n)$-integral by which $f(x)$ is integrable and $(\mathbf{A}, I_n)\int_0^1 f(x)dx = c$. This problem is not perfectly solved, but we get the following theorem.

Theorem 2. *If $f(x)$ of $\mathfrak{M}[0,1]$ satisfies $\int_0^1 f^+(x)dx = \int_0^1 f^-(x)dx$[2] $=\infty$ and c is given, there is $\mathfrak{H} \equiv \{h_n(x)\}$ such that $f(x)$ is $(\mathbf{A}, \mathfrak{H})$-integrable and $(\mathbf{A}, \mathfrak{H})\int_0^1 f(x)dx = c$.*

Proof. By the proof of Theorem 1, we have two sequences $\{I_m^+\}$ and $\{I_m^-\}$, of measurable sets and integers $m(\alpha)$ for $\alpha>0$, that satisfy $\lim_{m\to\infty} I_m^+ = 1$ and $\lim_{m\to\infty} I_m^- = 1$, $\mathrm{mes}\{(x\,;f^+(x)\geqq \alpha m)\cap I_m^+\}=0$ and $\mathrm{mes}\{(x\,;f^-(x)\geqq \alpha m)\cap I_m^-\}=0$ for $m>m(\alpha)$ and $\lim_{m\to\infty}\int_{I_m^+\cap I_m^-}[f(x)]_m dx$ exists. Put the limit c'. When c' is finite, put $d=c-c'$ and suppose $d>0$, then there are an integer m_1 and a measurable set J_{m_1} satisfying $J_{m_1}\subseteq I_{m_1}^-$ and $\int_{I_{m_1}^-}[f^-(x)]_{m_1}dx - d = \int_{J_{m_1}^-\setminus J_{m_1}}[f^-(x)]_{m_1}dx$. Take m_2 so large that $\int_{I_{m_2}^-\setminus I_{m_1}^-}[f^-(x)]_{m_2}dx > d$, there is a measurable set J_{m_2} satisfying $I_{m_2}^-\setminus I_{m_1}^- \supseteq J_{m_2}$ and $\int_{(I_{m_2}^-\setminus I_{m_1}^-)\setminus J_{m_2}}[f^-(x)]_{m_2}dx = \int_{I_{m_2}^-\setminus I_{m_1}^-}[f^-(x)]_{m_2}dx - d$. Continue this process, we can construct a sequence $\{J_{m_k}\}$. When $m_k\leqq n< m_{k+1}$, we define $I_n = I_n^+\cap(I_n^-\setminus J_{m_k})$ then $\lim_{n\to\infty}\int_{I_n}[f(x)]_n dx = c'-d=c$. When c' is infinite, we suppose $c'=\infty$ and $c>0$. There is an integer N satisfying $\int_{I_m^+}[f^+(x)]_m dx > \int_{I_m^-}[f^-(x)]_m dx$ for $m\geqq N$ then we can find a sequence $\{m_k\}$ satisfying $\int_{I_{m_{k-1}}^-}[f^-(x)]_{m_{k-1}}dx < \int_{I_{N+k}^+}[f^+(x)]_{N+k}dx - c \leqq \int_{I_{m_k}^-}[f^-(x)]_{m_k}dx$. Since $\int_{I_{m_{k-1}}^-}[f^-(x)]_{m_{k-1}}dx = \int_{I_{m_{k-1}}^-}[f^-(x)]_{m_k}dx$ for $m_k\geqq m(1)$, there is a sequence $\{I_{m_k}^-\}$ of measurable sets satisfying $\int_{I_{N+k}^+}[f^+(x)]_{N+k}dx - c = \int_{\tilde I_{m_k}^-}[f^-(x)]_{m_k}dx$ and $I_{m_{k-1}}\subseteq \tilde I_{m_k}^-\subseteq I_{m_k}^-$. If $n<N$, $h_n(x)=0$ and if $n\geqq N$, $h_n(x)=n$ in $I_n^+\setminus I_{m_{n-N}}^-$, $=m_{n-N}$ in $I_{m_{n-N}}^-$ and $=0$ otherwise. Then for sufficiently large n, $\int_{(x;|f(x)|\geqq \alpha h_n(x))} h_n(x)dx \leqq n\,\mathrm{mes}\{(x\,;f^+(x)\geqq \alpha n)\cap I_n^+\}$

2) $f^+(x)$ is the non-negative part of $f(x)$ and $f^-(x)=f^+(x)-f(x)$.

$+m_{n-N}\,\text{mes}\,\{(x\,;\,f^{-}(x)\geqq \alpha m_{n-N})\cap I^{-}_{m_{n-N}}\}=0$ and $\int_0^1 [f(x)]_{h_n}dx=c.$

(Q.E.D.)

Corollary. *If a function is integrable by any* $(\mathbf{A}, I_n)$*-integral, it is Lebesgue integrable.*

We get a theorem which is an extension of the Lebesgue's convergence theorem.

Theorem 3. *When a sequence* $\{f_m(x)\}$ *of* $(\mathbf{A}, \mathfrak{H})$*-integrable functions converges to* $f(x)$ *in measure, if* (1) $\int_{(x;|f_m(x)|\geqq \alpha h_n(x))} h_n(x)dx = o(1)$ *uniformly in* $m\geqq 0$ *for each* $\alpha>0$ *and* (2) $\lim_{n\to\infty}\int_0^1 [f_m(x)]_{h_n}dx = (\mathbf{A}, \mathfrak{H})\int_0^1 f_m(x)dx$ *uniformly in* $m\geqq 0$, *then* $f(x)$ *is* $(\mathbf{A}, \mathfrak{H})$*-integrable and* $(\mathbf{A}, \mathfrak{H})\int_0^1 f(x)dx=\lim_{m\to\infty}(\mathbf{A}, \mathfrak{H})\int_0^1 f_m(x)dx.$

D.E. Menshov showed, in [5], that for any function of $\mathfrak{M}[0,1]$ there exists a trigonometric series which converges to the function almost everywhere. So we need to consider the relation between Fourier coefficients and integrals. Let $\{\phi_n(x)\}_{n=1,2,\dots}$ be an orthonormal system in $[0,1]$. And we assume that $\phi_0(x)\equiv 1$ and for any l and m there is n satisfying $\phi_l(x)\phi_m(x)=\phi_n(x)$ almost everywhere.

Theorem 4. *If* $\sum_{n=0}^{\infty} c_n\phi_n(x)=f(x)$ *in measure, for any* $\alpha>0$ $\int_{(x;|S_m(x)|\geqq \alpha h_k(x))} h_k(x)dx=o(1)$ *uniformly in* $m\geqq 0$ *and* $\int_{(x;|S_m(x)|\geqq h_k(x))} \times S_m(x)\phi_n(x)dx=o(1)$ *uniformly in* $m\geqq n$, *where* $S_m(x)=\sum_{k=0}^{m} c_k\phi_k(x)$, *then* $f(x)\phi_n(x)$ *is* $(\mathbf{A}, \mathfrak{H})$*-integrable and* $c_n=(\mathbf{A}, \mathfrak{H})\int_0^1 f(x)\phi_n(x)dx.$

Corollary. $\sum_{n=0}^{\infty} c_n\phi_n(x)=0$ *in measure and* $\int_{(x;|S_m(x)|\geqq h_k(x))} S_m(x)\phi_n(x)dx = o(1)$ *uniformly in* $m\geqq n$, *then* $c_n=0$.

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