35. The Product of M-Spaces need not be an M-Space

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The notion of M-spaces has been introduced by K. Morita [1] and from that time on many interesting properties of M-spaces have been obtained by Morita and others. But a still unsolved problem is whether the cartesian product of M-spaces must be an M-space. The purpose of this note is to answer this question negatively.

We assume that all spaces are completely regular T_1-spaces and denote by βX and vX the Stone-Čech compactification and the Hewitt realcompactification of X respectively [2]. If X is not pseudocompact, then it is well known that $\beta X - vX \neq \varnothing$. A space X is said to be an *M-space* if there exists a normal sequence $\{\mathfrak{A}_n\,;\ n=1, 2, \cdots\}$ of open coverings of X satisfying the condition (M) below:

(M) If $\{K_n\}$ is a sequence of non-empty subsets of X such that $K_{n+1} \subset K_n$, $K_n \subset \mathrm{St}(x_0, \mathfrak{A}_n)$ for each n and for some fixed point x_0 of X, then $\cap \bar{K}_n \neq \varnothing$.

Theorem 1. *Suppose that X is not pseudocompact and P and Q are disjoint non-empty subsets of $\beta X - X$. If $X \cup P$ and $X \cup Q$ are countably compact, then $A \times B$ is not an M-space where $A = X \cup P \cup \{x^*\}$, $B = X \cup Q \cup \{x^*\}$ and x^* is an arbitrary point contained in $\beta X - vX$.*

Proof. Since A and B are countably compact these spaces are M-spaces. x^* belongings to $\beta X - vX$, there exists a continuous function f on βX such that $f > 0$ on X and $f(x^*) = 0$. It is obvious that

$$\cup (X \cap Z_n) = X \quad \text{where} \quad Z_n = \{x\,;\ f(x) \geqq 1/n,\ x \in \beta X\}.$$

Now suppose that $A \times B$ is an M-space. Then there exists a normal sequence $\{\mathfrak{A}_n\,;\ n=1, 2, \cdots\}$ of open coverings of $A \times B$ satisfying the condition (M). Let us put $s^* = (x^*, x^*)$. Since $\mathrm{St}(s^*, \mathfrak{A}_n)$ is an open set of $A \times B (\subset \beta X \times \beta X)$, there is an open set U_n (in βX) containing x^* such that

$$U_n \cap Z_n = \varnothing, \qquad \mathrm{cl}_{\beta X} U_{n+1} \subset U_n$$
$$\text{and} \quad (A \times B) \cap (U_n \times U_n) \subset \mathrm{St}(s^*, \mathfrak{A}_n).$$

As is well known every point of $\beta X - X$ is not G_δ in βX and hence $\cap U_n$ contains a point $y^* (\neq x^*)$ of $\beta X - X$ (notice that $\cup Z_n \supset X$ and $U_n \cap Z_n = \varnothing$). $x^* \neq y^*$ leads to the existence of an open set V of βX containing y^* whose closure does not contain x^*. We denote by $\varDelta(X)$ the diagonal set of $X \times X$ and by K_n the following set

$$(V \times V) \cap (U_n \times U_n) \cap \varDelta(X).$$

By the methods of construction of U_n and V, $U_n \cap V$ is an open set containing y^* and $K_n \neq \varnothing$, $K_{n+1} \subset K_n (n=1, 2, \cdots)$. Thus we have

$$K_n \subset \mathrm{St}(S^*, \mathfrak{A}_n) \quad \text{for each } n.$$

On the other hand, the fact that $Z_n \cap U_n = \varnothing$, $\cup Z_n \supset X$ and $A \cap B = \{x^*\}$ implies the following equality

$$\cap \mathrm{cl}_{\beta X \times \beta X}\{(U_n \times U_n) \cap \Delta(X)\} = \{s^*\}.$$

This shows that $\cap \bar{K}_n$ is empty which is a contradiction.

The countable compactness of $X \cup P (X \cup Q$ resp.) is necessary only to make sure the countable compactness of $A (B$ resp), consequently $A (B$ resp.) being an M-space. Thus from our method of proof it is easy to see the following

Corollary 1. *Suppose that X is not pseudocompact and $P, Q \subset \beta X - X$. If $X \cup P$ and $X \cup Q$ are M-spaces and if $\beta X - \upsilon X$ contains a point x^* such that $x^* \in P \cap Q$ and there exists an open set U of βX containing x^* with $U \cap P \cap Q = \{x^*\}$, then $(X \cup P) \times (X \cup Q)$ is not an M-space.*

Example. Let N be the discrete space consisting of positive integers. Novák [3] has proved that there are subsets P and Q in $\beta N - N$ such that $P \cap Q = \varnothing$, $P \cup Q = \beta N - N$ and both spaces $N \cup P$ and $N \cup Q$ are countably compact and have no infinite compact subsets. It is obvious that these spaces N, P and Q are sets satisfying the assumption desired in Theorem 1.

Remark. A point x of X is said to be a *q-point* if it has a sequence of neighborhoods N_i such that $x_i \in N_i$ and the x_i are all distinct, then $x_1, x_2, \cdots$ has an accumulation point in X. A space X is called to be a *q-space* if every point of X is a q-point [5]. The proof above implies that the point s^* is not a q-point.

Recently T. Ishii, S. Tsuda, and S. Kunugi [4] considered the class $\mathcal{C}$ of all spaces X such that there exists a normal sequence $\{\mathfrak{A}_n ; n=1, 2, \cdots\}$ of open coverings of X satisfying the condition $(*)$: If $\{x_n\}$ is a sequence of points of X such that $x_n \in \mathrm{St}(x_0, \mathfrak{A}_n)$ for each n and for some fixed point x_0 of X, then there exists a subsequence $\{x_{n_i} ; i=1, 2, \cdots\}$ which has the compact closure. They proved that if X belongs to $\mathcal{C}$, then the product $X \times Y$ is an M-space for every M-space Y. From this fact and our Theorem 1, we have

Corollary 2. *If X is not pseudocompact and A is a countably compact space which belongs to $\mathcal{C}$ and $X \subset A \subset \beta X$, then $X \cup Q$ is not countably compact for any subset Q of $\beta X - A$.*

References

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