33. *On Certain Mixed Problem for Hyperbolic Equations of Higher Order*

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1. Introduction. Let Ω be the half-space of $\boldsymbol{R}^n$: $\{(x_1, x_2, \cdots, x_n) \mid x_n > 0\}$, and Γ be a boundary of Ω.

Consider the hyperbolic equation

$$(1.1)\quad Lu = \left(\frac{\partial^{2m}}{\partial t^{2m}} + a_1(x, D)\frac{\partial^{2m-1}}{\partial t^{2m-1}} + \cdots + a_{2m}(x, D)\right)u + B\left(x, D, \frac{\partial}{\partial t}\right)u = f$$

where $a_k(x, D) = \sum_{|\alpha|=k} a_\alpha(x) D^\alpha$, $D_j = \frac{1}{\sqrt{-1}}\frac{\partial}{\partial x_j}$, $\alpha = (\alpha_1, \cdots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, and B is an arbitrary differential operator of order $(2m-1)$.

We assume that all coefficients are sufficiently differentiable and bounded with their derivatives in $\boldsymbol{R}^n$.

Our aim of the present note is to assert the following

Theorem 1. *We assume that* $a_{\alpha_1 \cdots \alpha_n}(x', 0) = 0$ *when* α_n *is odd. Let all the roots* $\tau_i(x, \xi)$, $(i = 1, \cdots, 2m)$ *with respect to* τ *of the equation* $\tau^{2m} + a_1(x, \xi)\tau^{2m-1} + \cdots + a_{2m}(x, \xi) = 0$ *be pure imaginary, distinct and not zero*, uniformly. *Then for any* $f(t, x) \in C^1([0, T]; L^2(\Omega))$ *and any initial data* $\left(u(0, x), \frac{\partial u}{\partial t}(0, x), \cdots, \frac{\partial^{2m-1} u}{\partial t^{2m-1}}(0, x)\right) \in \mathcal{D}_i$ $(i = 1, 2)$, *there exists a unique solution* u *of the equation* (1.1) *satisfying boundary conditions*

$$(1.2)\qquad u|_\Gamma = \Delta u|_\Gamma = \cdots = \Delta^{m-1} u|_\Gamma = 0,$$

or

$$(1.3)\qquad \frac{\partial}{\partial x_n} u|_\Gamma = \frac{\partial}{\partial x_n}\Delta u|_\Gamma = \cdots = \frac{\partial}{\partial x_n}\Delta^{m-1} u|_\Gamma = 0.$$

The solution satisfies $\left(u(t, x), \frac{\partial u}{\partial t}(t, x), \cdots, \frac{\partial^{2m} u}{\partial t^{2m}}(t, x)\right) \in C^0([0, T]; \mathcal{D}_i \times L^2(\Omega))$, *where* $\mathcal{D}_1 = D(\Lambda_-^{2m}) \times \cdots \times D(\Lambda_-)$, $\mathcal{D}_2 = D(\Lambda_+^{2m}) \times \cdots \times D(\Lambda_+)$. In the case of Dirichlet type boundary condition (1.2), we consider $\mathcal{D}_1$, and in the case of Neumann type boundary condition (1.3), we consider $\mathcal{D}_2$. The definitions of Λ_+, Λ_- are represented in the following section.

It is not difficult to show that from the considerations in the proof of Theorem 1 it implies the theorems obtained by S. Mizohata [5] and by S. Miyatake [4]. The method of proof of Theorem 1 is based on singular integral operators with boundary conditions developed below and on Leray's one [3].

The detailed treatment and other interesting results shall be published elsewhere.

2. Singular integral operators with boundary conditions.

Definition 1. Let $A(\xi)$ be any bounded function in $\boldsymbol{R}^n$, homogeneous of degree zero. For $u(x)\in L^2(\boldsymbol{R}^n_+)$, we define

$$A(D)u\equiv F^{+\prime}(A(\xi)F^+u(\xi)),\quad A_2(D)u\equiv F^{-\prime}(A(\xi)F^-u(\xi)),$$

where
$$(F^+u)(\xi)=\int_{-\infty}^{\infty}\int_0^{\infty}e^{-ix'\xi'}\cos(x_n\xi_n)u(x',x_n)dx'dx_n,$$
$$(F^-u)(\xi)=\int_{-\infty}^{\infty}\int_0^{\infty}e^{-ix'\xi'}\sin(x_n\xi_n)u(x',x_n)dx'dx_n,$$
$$(F^{+\prime}u)(\xi)=\frac{1}{(2\pi)^{n-1}}\cdot\frac{2}{\pi}\int_{-\infty}^{\infty}\int_0^{\infty}e^{ix'\xi'}\cos(x_n\xi_n)u(x',x_n)dx'dx_n,$$
$$(F^{-\prime}u)(\xi)=\frac{1}{(2\pi)^{n-1}}\cdot\frac{2}{\pi}\int_{-\infty}^{\infty}\int_0^{\infty}e^{ix'\xi'}\sin(x_n\xi_n)u(x',x_n)dx'dx_n,$$
$$\boldsymbol{R}^n\ni x=(x_1,\cdots,x_{n-1},x_n)=(x',x_n),$$
$$\boldsymbol{R}^n_+=\{x\in\boldsymbol{R}^n\,;\,x_n>0\}=\Omega,$$
$$x'\xi'=\sum_{j=1}^{n-1}x_j\xi_j,\quad i=\sqrt{-1}.$$

Definition 2. We define the following positive self-adjoint operators in $L^2(\boldsymbol{R}^1_+)$ or $L^2(\boldsymbol{R}^1)$: we set

$$H^2_+=-\frac{d^2}{dx^2},\quad D(H^2_+)=\left\{u\in H^2(\boldsymbol{R}^1_+)\,;\,\frac{du}{dx}(0)=0\right\},$$
$$H^2_-=-\frac{d^2}{dx^2},\quad D(H^2_-)=H^2(\boldsymbol{R}^1_+)\cap H^1_0(\boldsymbol{R}^1_+)$$
$$H^2=-\frac{d^2}{dx^2}.\quad D(H^2)=H^2(\boldsymbol{R}^1),$$

and set $H_+=(H^2_+)^{\frac12}$, $H_-=(H^2_-)^{\frac12}$, $H=(H^2)^{\frac12}$. Then we have that $D(H_+)=H^1(\boldsymbol{R}^1_+)$, $D(H_-)=H^1_0(\boldsymbol{R}^1_+)$, $D(H)=H^1(\boldsymbol{R}^1)$.

Definition 3. We set

$$\Lambda_+=-(\Delta'+H^2_{+},)^{\frac12},\quad \Lambda_-=(-\Delta'+H^2_-)^{\frac12},\quad \Lambda=(-\Delta'+H^2)^{\frac12},$$
$$D(\Lambda_+)=H^1(\boldsymbol{R}^n_+),\quad D(\Lambda_-)=H^1_0(\boldsymbol{R}^n_+),\quad D(\Lambda)=H^1(\boldsymbol{R}^n),$$
$$\text{where}\quad \Delta'=\sum_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}.$$

It follows that

$$\Lambda_+u=\Lambda\tilde{u}|_{x_n>0}\ \text{for}\ u(x)\in D(\Lambda_+),\quad \Lambda_-u=\Lambda\tilde{\tilde{u}}|_{x_n>0}\ \text{for}\ u(x)\in D(\Lambda_-),$$

where
$$\tilde{u}(x',x_n)=\begin{cases}u(x',x_n) & \text{for}\ x_n>0,\\ u(x',-x_n) & \text{for}\ x_n<0,\end{cases}$$
$$\tilde{\tilde{u}}(x',x_n)=\begin{cases}u(x',x_n) & \text{for}\ x_n>0\\ -u(x',-x_n) & \text{for}\ x_n<0.\end{cases}$$

In what follows we consider only Λ_+, as we can consider Λ_- similar to Λ_+.

Definition 4. $a(x,\xi)\in \Xi_4^\infty$ means that

$$a(x,\xi)\in C^{4,\infty}_{x,\xi}(\overline{R^n_+}\times(R^n-\{0\})),\quad a(x,\lambda\xi)=a(x,\xi)\quad\text{for}\quad \lambda>0,$$

and for every integer $s(\geq 0)$, there exists $M_s(a)$ $(<\infty)$ such that

$$\sum_{\substack{|\mu|\leq 4\\|\nu|\leq s}}\sup_{\substack{|\xi|=1\\x\in\overline{R}^n_+}}\left|\left(\frac{\partial}{\partial x}\right)^{\mu}\left(\frac{\partial}{\partial\xi}\right)^{\nu}a(x,\xi)\right|\leq M_s(a).$$

Theorem 2. *Let $a(x,\xi)$, $b(x,\xi)\in\Xi_4^\infty$. We set singular integral operators $a(x,D)$, $b(x,D)$ with symbol $a(x,\xi)$, $b(x,\xi)$, respectively, that is, for $u(x)\in L^2(R^n_+)$, $a(x,D)u=F^{+\prime}(a(x,\xi)F^+u(\xi))$. Then, for $u(x)\in D(\Lambda_+)$, we obtain the following estimates.*

i) $\|(a(x,D)b(x,D)-b(x,D)a(x,D))\Lambda_+u\|_{x_n>0}$
$\leq c(M_{2([\frac{3}{2}n]+3)}(a)\cdot M_{2(n+1)}(b)+M_{2(n+1)}(a)M_{2([\frac{3}{2}n]+3)}(b))\|u\|_{x_n>0}$,

ii) $\|(a(x,D)\Lambda_+-\Lambda_+a(x,D))u\|_{x_n>0}\leq cM_{2(n+1)}(a)\|u\|_{x_n>0}$,

iii) $\|(a(x,D)^*-a^{\#}(x,D))\Lambda_+u\|_{x_n>0}\leq cM_{2([\frac{3}{2}n]+3)}(a)\|u\|_{x_n>0}$,

iv) $\|(a(x,D)b(x,D)-(a\circ b)(x,D))\Lambda_+u\|_{x_n>0}$
$\leq cM_{2([\frac{3}{2}n]+3)}(a)M_{2(n+1)}(b)\|u\|_{x_n>0}$.

Here $\|u\|^2_{x_n>0}=\int_{R^n_+}|u|^2dx$, $a^{\#}(x,D)$, $(a\circ b)(x,D)$ are singular integral operators with symbol $\overline{a(x,\xi)}$, $a(x,\xi)b(x,\xi)$, respectively, c depends only on dimension n and [ ] denotes the Gauss symbol.

Definition 5. $\mathfrak{A}$ is the algebra generated by $a(x,\xi)\in\Xi_4^\infty$ with the property: $a(x,\xi',\xi_n)=a(x,\xi',-\xi_n)$ and $f(x)\frac{\xi_n}{|\xi|}$ with the property: $f(x',0)=0$ and $f(x)\in C^4(\overline{R}^n_+)$. For $\alpha(x,\xi)=\sum_{i=1}^m a_i(x,\xi)f_i(x)\frac{\xi_n}{|\xi|}$, we associate with the singular integral operator $\alpha(x,D)$ as follows:

$$\alpha(x,D)u=\sum_{i=1}^m F'(f_i(x)\frac{\xi_n}{|\xi|}a_i(x,\xi)F\tilde{u})|_{x_n>0},\quad\text{for}\quad u\in L^2(R^n_+),$$

where F is Fourier transformation and F' its inverse.

Theorem 3. *For the symbols $\alpha(x,\xi)$, $\beta(x,\xi)\in\mathfrak{A}$, the statements of Theorem 2 are also valid.*

The proof of Theorem 1 is a direct consequence of Theorem 3 from which it is seen that the proof is accomplished by the familiar method with the use of singular integral operator with respect to the Cauchy problem for hyperbolic operators ([1], [3], [5]-[7]).

References

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