32. *Mappings and M-Spaces*

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Let us recall some of the interesting theorems on metric spaces and compact spaces in relation with maps (=mappings). (As for the references and proofs of these theorems as well as terminologies and symbols, see J. Nagata [4] and [5]. All spaces are at least Hausdorff, and all maps are continuous in the present paper unless the contrary is explicitly mentioned.)

1. A T_1-space, not necessarily Hausdorff, is the image of a metric space by an open continuous map iff (=if and only if) it is 1-st countable (V. Ponomarev-S. Hanai).

2. Every metric space with weight $|A|$ (=the cardinality of the set A) is the image of a subset of Baire's 0-dimensional space $N(A)$ (=the product of countably many copies of the discrete space A) by a perfect map. (K. Morita)

3. Every compact (Hausdorff) space with weight $|A|$ is the continuous image of a closed set of the cantor discontinuum $D(A)$. (P. S. Alexandroff)

4. Every metric space with weight $|A|$ is homeomorphic to a subset of generalized Hilbert space $H(A)$. (C. H. Dowker)

5. Every compact space with weight $|A|$ is homeomorphic to a closed subset of the product of the copies I_α, $\alpha \in A$, of the unit interval $[0, 1]$. (*A*. Tychonoff-P. Urysohn)

As well known, the concept of M-space (paracompact M-space) is an important generalization of that of metric space as well as countably compact space (compact space). Therefore it is natural to try to extend the above theorems to M-spaces and paracompact M-spaces. The purpose of the present paper is to continue our study along this line which started in our previous paper [5].

Theorem 1. *A regular space Y is a q-space in the sense of E. Michael* [1] *iff there are an M-space X and a continuous open* map f *from X onto Y.*

Proof. Sufficiency directly follows from the condition satisfied by X and Y by use of Lemma 1 of [5]. To prove necessity we should note that a regular space is a q-space iff each point has a sequence $U_1, U_2, \cdots$ of open nbds (=neighborhoods) such that $U_1 \supset \bar{U}_2 \supset U_2 \supset \bar{U}_3 \supset \cdots$ and such that if $x_i \in U_i$, $i=1, 2, \cdots$, then $\{x_i \mid i=1, 2, \cdots\}$

has a cluster point. Such a sequence of nbds was called a q-sequence of nbds in [5].

Now, let Y be a regular q-space, $\{U_\alpha \mid \alpha \in A\}$ the collection of all open sets of Y, and $\{V_\lambda \mid \lambda \in \Lambda\}$ the collection of all binary open covers of Y. To each open cover $V_\lambda = \{V, V'\}$ we associate a cover V'_λ consisting of $V - V'$, $V' - V$ and $V \cap V'$ which we denote by V_1^λ, V_2^λ and V_3^λ respectively. Let us denote by 3^Λ the product space of the spaces T_λ, $\lambda \in \Lambda$, consisting of three points a_1^λ, a_2^λ and a_3^λ, whose topology (=the set of all open sets) consists of $\varnothing$, $\{a_3^\lambda\}$, $\{a_1^\lambda, a_3^\lambda\}$, $\{a_2^\lambda, a_3^\lambda\}$ and $\{a_1^\lambda, a_2^\lambda, a_3^\lambda\}$. Note that each point of $N(A) \times 3^\Lambda$, where $N(A)$ denotes Baire's 0-dimensional space, can be represented as $(\alpha_1, \alpha_2, \cdots) \times (a_{i(\lambda)}^\lambda \mid \lambda \in \Lambda)$, which we may abridge to $\alpha \times (a_{i(\lambda)}^\lambda)$ or $(\alpha_1, \alpha_2 \cdots) \times a$. Let us define a subset X of $N(A) \times 3^\Lambda$ by

$$X = \left\{ (\alpha_1, \alpha_2, \cdots) \times (a_{i(\lambda)}^\lambda \mid \lambda \in \Lambda) \mid U_{\alpha_1}, U_{\alpha_2}, \cdots \text{ form a } q\text{-sequence of nbds in } Y \text{ such that } \left[\bigcap_{i=1}^\infty U_{\alpha_i}\right] \cap \left[\bigcap_{\lambda \in \Lambda} V_{i(\lambda)}^\lambda\right] \neq \varnothing \right\}.$$

We define a map $f \mid X \to Y$ by

$$f((\alpha_1, \alpha_2, \cdots) \times (a_{i(\lambda)}^\lambda \mid \lambda \in \Lambda)) = \left[\bigcap_{i=1}^\infty U_{\alpha_i}\right] \cap \left[\bigcap_{\lambda \in \Lambda} V_{i(\lambda)}^\lambda\right].$$

Theorem 2. *Every paracompact M-space Y with weight $|A|$ is the image of a 0-dimensional, paracompact M-space X by a perfect map, where X is a closed subset of the product of $D(A)$ and a subset of $N(A)$.*

Proof. Let $\{V_\alpha \mid \alpha \in A\}$ be an open basis of Y. Put $\mathcal{V}_\alpha = \{\bar{V}_\alpha, X - V_\alpha\}$. We denote this closed cover by $V_\alpha = \{G_1^\alpha, G_2^\alpha\}$. On the other hand, Y has a sequence $\mathcal{U}_1, \mathcal{U}_2, \cdots$ of open covers such that $\mathcal{U}_1 > \mathcal{U}_2^* > \mathcal{U}_2 > \mathcal{U}_3^* > \cdots$ and such that $\{S(y_0, \mathcal{U}_i) \mid i = 1, 2, \cdots\}$ is a q-sequence of nbds of y_0 for each $y_0 \in Y$. Now put $\mathcal{F}_i = \{\bar{U} \mid U \in \mathcal{U}_i\}$. Then $\{S(y_0, \mathcal{F}_i) \mid i = 1, 2, \cdots\}$ is a q-sequence; i.e., if $y_i \in S(y_0, \mathcal{F}_i)$, $i = 1, 2, \cdots$, then $\{y_i\}$ has a cluster point in $\bigcap_{i=1}^\infty S(y_0, \mathcal{F}_i)$. Observe that each $\mathcal{F}_i$ has at most $|A|$ elements because it is locally finite. We may assume that each element of $\mathcal{F}_i$ is a union of elements of $\mathcal{F}_{i+1}$, because otherwise we can replace $\mathcal{F}_{i+1}$ with $\mathcal{F}_i \wedge \mathcal{F}_{i+1}$. Thus we may put

$$\mathcal{F}_i = \{F(\alpha_1 \cdots \alpha_i) \mid \alpha_1, \cdots \alpha_i \in A\}$$

assuming that $F(\alpha_1 \cdots \alpha_i) = \cup \{F(\alpha_1 \cdots \alpha_i \alpha_{i+1}) \mid \alpha_{i+1} \in A\}$, where some $F(\alpha_1 \cdots \alpha_i)$ may be empty, and $F(\alpha_1 \cdots \alpha_i) = F(\alpha_1 \cdots \alpha_i)$ may happen for $(\alpha_1 \cdots \alpha_i) \neq (\alpha'_1 \cdots \alpha'_2)$. However, note that for a fixed i, any point of Y has a nbd which intersects $F(\alpha_1, \cdots, \alpha_i)$ only for finitely many different $(\alpha_1 \cdots \alpha_i)$'s. When we talk of the local finiteness of $\mathcal{F}_i$ in the following discussion, we mean the one in such a strong sense. Also remember the following observation (0) which will be used later.

(0) If $\bigcap_{k=1}^{\infty} F(\alpha_1 \cdots \alpha_k)=C\neq\varnothing$, and if $C\cap H=\varnothing$ for a closed set H, then $F(\alpha_1 \cdots \alpha_k)\cap H=\varnothing$ for some k. C is a compact set. Because $\{F(\alpha_1, \cdots, \alpha_k) \mid k=1, 2, \cdots\}$ is a q-sequence and Y is paracompact.

Now, we define a subset S of $N(A)$ by

$$S=\left\{(\alpha_1, \alpha_2, \cdots) \mid (\alpha_1, \alpha_2, \cdots)\in N(A) \bigcap_{k=1}^{\infty} F(\alpha_1, \cdots, \alpha_k)\neq\varnothing\right\}$$

and a subset X of $S\times D(A)$ by

$$X=\left\{(\alpha_1, \alpha_2, \cdots)\times(b^{\lambda}_{i(\lambda)} \mid \lambda\in A)\in S\times D(A) \mid \left[\bigcap_{k=1}^{\infty} F(\alpha_1, \cdots \alpha_k)\right] \cap\left[\bigcap_{\lambda\in A} G^{\lambda}_{i(\lambda)}\right]\neq\varnothing\right\}.$$

(Note that every point of $N(A)\times D(A)$ can be represented as $(\alpha_1, \alpha_2, \cdots)\times(b^{\lambda}_{i(\lambda)} \mid \lambda\in A)$ with $\alpha_1, \alpha_2, \cdots\in A$ and a function $i \mid A\to\{1, 2\}$ assuming that $D(A)$ is the product of the two point discrete spaces $D_\lambda=\{b^{\lambda}_1, b^{\lambda}_2\}$, $\lambda\in A$.)

Then we naturally define a map $f \mid X\to Y$ by

$$f((\alpha_1, \alpha_2, \cdots)\times(b^{\lambda}_{i(\lambda)} \mid \lambda\in A))=\left[\bigcap_{k=1}^{m} F(\alpha_1 \cdots \alpha_k)\right]\cap\left[\bigcap_{\lambda\in A} G^{\lambda}_{i(\lambda)}\right].$$

It is easy to see that f is a continuous, onto map, so it is left to the reader.

Let us first prove that X is a closed set of $S\times D(A)$. Let

$$p=(\alpha_1, \alpha_2, \cdots)\times(b^{\lambda}_{i(\lambda)} \mid \lambda\in A)\in S\times D(A)-X.$$

Then

$$C=\bigcap_{k=1}^{\infty} F(\alpha_1 \cdots \alpha_k)\neq\varnothing, \qquad\text{and}\qquad C\cap\left[\bigcap_{\lambda\in A} G^{\lambda}_{i(\lambda)}\right]=\varnothing.$$

Since C is compact,

$$C\cap\left[\bigcap_{l=1}^{m} G^{\lambda_l}_{i(\lambda_l)}\right]=\varnothing \qquad\text{for some } \lambda_1, \cdots, \lambda_m.$$

Therefore, by (0)

$$F(\alpha_1 \cdots \alpha_k)\cap\left[\bigcap_{l=1}^{m} G^{\lambda_l}_{i(\lambda_l)}\right]=\varnothing \qquad\text{for some } k.$$

Thus

$$\{(\beta_1, \beta_2, \cdots)\times(b^{\lambda}_{j(\lambda)} \mid \lambda\in A)\in S\times D(A) \mid \beta_1=\alpha_1, \cdots, \beta_k=\alpha_k;\ j(\lambda_1)=i(\lambda_1), \cdots, j(\lambda_m)=i(\lambda_m)\}$$

is a nbd of p which is disjoint from X. This proves that X is closed in $S\times D(A)$.

Since S is metric, $D(A)$ is compact and both are 0-dimensional, $S\times D(A)$ is a 0-dimensional paracompact M-space (See K. Morita, [2] and [3]) and hence X is also a 0-dimensional paracompact M-space.

Let $y\in Y$; then $f^{-1}(y)$ can be proved to be countably compact, and therefore compact. But we are going to prove a stronger assertion as follows. Choose $(\alpha_1, \alpha_2 \cdots)$ such that $y \in \operatorname{Int} F(\alpha_1 \cdots \alpha_j)$, $j=1, 2, \cdots$. Note that $\{F(\alpha_1 \cdots \alpha_j) \mid j=1, 2, \cdots\}$ is a q-sequence. Suppose $\gamma^j \in f^{-1}(F(\alpha_1 \cdots \alpha_j))$, $j=1, 2, \cdots$. Then we can prove that $[\gamma^j \mid j=1, 2, \cdots\}$ has a cluster point in X. This assertion implies that $\bigcap_{j=1}^{\infty} f^{-1}(F(\alpha_1 \cdots \alpha_j))$ and accordingly $f^{-1}(y)$, too, is compact.

Now, let $\gamma^j = (\alpha_1^j, \alpha_2^j, \cdots) \times (b_{i(j,\lambda)}^{\lambda} \mid \lambda \in A)$; then $f(\gamma^j) \in F(\alpha_1 \cdots \alpha_j)$, which implies that

$$F(\alpha_1^j, \cdots, \alpha_1^j) \cap F(\alpha_1, \cdots, \alpha_j) \neq \phi, \qquad j=1, 2, \cdots$$

Choose $y'_j \in F(\alpha_1^j, \cdots, \alpha_j^j) \cap F(\alpha_1, \cdots, \alpha_j)$. Then $\{y'_j \mid j=1, 2, \cdots\}$ has a cluster point y' in $\bigcap_{j=1}^{\infty} F(\alpha_1, \cdots, \alpha_j)$.

Since $\mathcal{F}_1$ is locally finite, y' has a nbd N which intersects $F(\alpha_1^j)$ for only finitely many different α_1^j. On the other hand $y'_j \in N$ for infinitely many different j. Therefore N intersects $F(\alpha_1^j)$ for infinitely many j. Hence we can choose a subsequence $j(11), j(12), j(13), \cdots$ of $\{1, 2, 3, \cdots\}$ such that

$$\alpha_1^{j(11)} = \alpha_1^{j(12)} = \alpha_1^{j(13)} = \cdots$$

Denote these same elements of A by B_1.

Similarly, we can choose a subsequence $j(21), j(22), j(23), \cdots$ of $\{j(1i) \mid i=1, 2, \cdots\}$ such that

$$(\alpha_1^{j(21)}, \alpha_2^{j(21)}) = (\alpha_1^{j(22)}, \alpha_2^{j(22)}) = (\alpha_1^{j(23)}, \alpha_2^{j(23)}) = \cdots$$

The first elements of these pairs are equal to β_1.

Denote the second elements by β_2.

Repeating the same process, we obtain a sequence $\beta_1, \beta_2, \beta_3, \cdots$ of elements of A satisfying

(B) $\qquad (\beta_1, \cdots, \beta_k) = (\alpha_1^{j(kk)}, \cdots, \alpha_k^{j(kk)}), \qquad k=1, 2, \cdots.$

Then

$$F(\beta_1, \cdots, \beta_k) = F(\alpha_1^{j(kk)}, \cdots, \alpha_k^{j(kk)}) \supset F(\alpha_1^{j(kk)}, \cdots, \alpha_{j(kk)}^{j(kk)}).$$

Therefore $F(\beta_1, \cdots, \beta_k) \cap F(\alpha_1, \cdots, \alpha_{j(kk)}) \neq \varnothing$. Since $\{F(\alpha_1, \cdots, \alpha_j) \mid j=1, 2, \cdots\}$ is a q-sequence,

$$\left[\bigcap_{k=1}^{\infty} F(\beta_1, \cdots, \beta_k)\right] \cap \left[\bigcap_{k=1}^{\infty} F(\alpha_1, \cdots, \alpha_{j(kk)})\right] \neq \varnothing.$$

This implies that $(\beta_1, \beta_2, \cdots) \in S$. Let $C = \bigcap_{k=1}^{\infty} F(\beta_1, \cdots, \beta_k)$; then C is a non-empty compact subset of Y.

Now, put $b_k = (b_{i(j(kk),\lambda)}^{\lambda} \mid \lambda \in A)$, $k=1, 2, \cdots$; then $\{\beta_k \mid k=1, 2, \cdots\}$ is a point sequence in the compact space $D(A)$, and hence it has a cluster point $b = (b_{i(\lambda)}^{\lambda} \mid \lambda \in A)$ in $D(A)$. Put $\gamma = (\beta_1, \beta_2, \cdots) \times b$; then $\gamma \in S \times D(A)$. Let $\lambda_1, \cdots, \lambda_l$ be arbitrary members of A and k_0 a natural number. Then there is $k \geq k_0$ such that b and b_k have the same coordinates for $\lambda_1, \cdots, \lambda_l$, i.e., $i(j(kk), \lambda_m) = i(\lambda_m)$ for $m=1, \cdots, l$. Thus from (B) it follows that γ is a cluster point of $\{\gamma^{j(kk)} \mid k=1, 2, \cdots\}$

and accordingly of $\{\gamma^j \mid j=1, 2, \cdots\}$ in $S\times D(A)$. Now observe that

$$f(\gamma^{j(kk)}) \in F(\beta_1, \cdots, \beta_k) \cap \left[\bigcap_{m=1}^{l} G_{i(\lambda m)}^{\lambda m}\right]$$

for k chosen in the above discussion. This proves that $\{F(\beta_1, \cdots, \beta_k) \mid k=1, 2, \cdots\} \cup \{G_{i(\lambda)}^{\lambda} \mid \lambda \in A\}$ forms a closed collection in Y with f.i.p. (finite intersection property). Thus the f.i.p. of $\{C, C_{i(\lambda)}^{\lambda} \mid \lambda \in A\}$ follows from (Q). Since C is compact, we obtain $C \cap [\cap \{G_{i(\lambda)}^{\lambda} \mid \lambda \in A\}] \neq \varnothing$. Therefore, $\gamma \in X$, proving our assertion, which implies the compactness of $f^{-1}(y)$ as a corollary.

Finally, let us prove that f is a closed map. Let G be a closed set in X and let $y \notin f(G)$ in Y. Choose $\{\alpha_1, \alpha_2, \cdots\}$ such that $y \in \bigcap_{k=1}^{\infty} \operatorname{Int} F(\alpha_1, \cdots, \alpha_k)$. Put $C=\bigcap_{k=1}^{\infty} F(\alpha_1, \cdots, \alpha_k)$. As proved before, $f^{-1}(C)$ is a compact set. Therefore, $C \cap f(G) = f(f^{-1}(C) \cap G)$ is compact, and hence y has a nbd V satisfying $\bar{V} \cap C \cap f(G) = \varnothing$. Now, assume $F(\alpha_1 \cdots \alpha_k) \cap \bar{V} \cap f(G) \neq \varnothing$, $k=1, 2, \cdots$. Then we can choose $y_k \in F(\alpha_1 \cdots \alpha_k) \cap \bar{V} \cap f(G)$ and $x_k \in f^{-1}(y_k) \cap G$. Then $x_k \in f^{-1}(F(\alpha_1 \cdots \cdots \alpha_k))$, and hence, as we proved before, $\{x_k \mid k=1, 2, \cdots\}$ has a cluster point $x \in G$. Thus, $f(x) \in f(G)$ is a cluster point of $\{y_k \mid k=1, 2, \cdots\}$ in Y. Hence $f(x) \in C \cap \bar{V} \cap f(G)$, which is a contradiction. Therefore $f(\alpha_1 \cdots \alpha_k) \cap \bar{V} \cap f(G) = \varnothing$ for some k. Namely, $\operatorname{Int} F(\alpha_1 \cdots \alpha_k) \cap V$ is a nbd of y which is disjoint from $f(G)$ proving that $f(G)$ is a closed set. This completes our proof that f is a perfect map.

By a somewhat similar discussion we can prove the following theorem.

Theorem 3. *A space X with weight $|A|$ is a paracompact M-space iff it is homeomorphic to a closed subset of $S\times P(A)$, where S is a subspace of generalized Hilbert space $H(A)$, and $P(A)$ is the product of the copies I_α, $\alpha \in A$, of the unit interval* $[0, 1]$.

Problem. Is every T_1, M-space homeomorphic to a closed subset of the product of a metric space and a countably compact space?

References

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