53. *On Ranked Spaces and Linearity. II*

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(Comm. by Kinjirô KUNUGI, M. J. A., April 12, 1969)

In this note we shall give a definition of linear ranked spaces, axioms of which are weaker than those given in [2]. Sometimes this definition is more convenient to use, in particular, to study for the notions connected with fundamental sequences of neighbourhoods. Hereafter we shall treat only ranked spaces with indicator ω_0[1]. Throughout this note, $x, y, \cdots$ will denote points of a ranked space, $\mathfrak{B}_n(x)$ the system of neighbourhoods of x with rank n, $\{u_n(x)\}$, $\{v_n(x)\}$, $\cdots$ fundamental sequences of neighbourhoods with respect to x.

§1. Definition of linear ranked spaces. Let E be a ranked space, and also a linear space over real or complex field. We call E a linear ranked space, if linear operations in E are continuous in the following sense:

(I) For any $\{u_n(x)\}$ and $\{v_n(y)\}$, there is a $\{w_n(x+y)\}$ such that $u_n(x)+v_n(y)\subseteq w_n(x+y)$.

(II) For any $\{u_n(x)\}$ and $\{\lambda_n\}$ with $\lim \lambda_n=\lambda$, there is a $\{v_n(\lambda x)\}$ such that $\lambda_n u_n(x)\subseteq v_n(\lambda x)$.

(I) implies the continuity of addition; if $\{\lim x_n\}\ni x$ and $\{\lim y_n\}\ni y$, then $\{\lim (x_n+y_n)\}\ni x+y$, and (II), the continuity of scalar multiplication; if $\{\lim x_n\}\ni x$ and $\lim \lambda_n=\lambda$, then $\{\lim \lambda_n x_n\}\ni \lambda x$.

§2. The neighbourhoods of zero. Let E be a linear ranked space. We will denote the system of neighbourhoods of 0 with rank n by $\mathfrak{B}_n$, and fundamental sequences with respect to 0 by $\{U_n\}$, $\{V_n\}$, $\cdots$. Obviously $\{\mathfrak{B}_n\}$ satisfies the axioms (A), (B), (a), (b) in [2].

Furthermore, from (I), (II), we get following properties.

(RL_1) For any $\{U_n\}$ and $\{V_n\}$, there is a $\{W_n\}$ such that $U_n+V_n\subseteq W_n$.

(RL_2) (i) For any $\{U_n\}$ and λ, there is a $\{V_n\}$ such that $\lambda U_n\subseteq V_n$.

(ii) For any x and $\{\lambda_n\}$ with $\lim \lambda_n=0$, there is a $\{V_n\}$ such that $\lambda_n x\in V_n$.

(iii) For any $\{U_n\}$ and $\{\lambda_n\}$ with $\lim \lambda_n=0$, there is a $\{V_n\}$ such that $\lambda_n U_n\subseteq V_n$.

(RL_3) Let x be any point in E. For any $\{U_n\}$ there is a $\{v_n(x)\}$ such that $x+U_n\subseteq v_n(x)$, and, conversely, for any $\{u_n(x)\}$ there is a $\{V_n\}$ such that $u_n(x)\subseteq x+V_n$.

Proof. (RL_1), (RL_2) (i), (iii) are immediate consequences of (I), (II), respectively, putting $x=y=0$, $\lambda_n=\lambda$, or $\lambda=0$. As for (RL_2) (ii), taking some $\{u_n(x)\}$ and applying (II) for $\lambda=0$, there is a $\{V_n\}$ such that $\lambda_n u_n(x)\subseteq V_n$. Since $x\in u_n(x)$, we have $\lambda_n x\in V_n$. Now, we shall show (RL_3). Take any $\{U_n\}$ and some $\{u_n(x)\}$. From (I), there is a $\{v_n(x)\}$ such that $u_n(x)+U_n\subseteq v_n(x)$, and therefore $x+U_n\subseteq v_n(x)$. Conversely, for any $\{u_n(x)\}$, taking some $\{v_n(-x)\}$, there is a $\{V_n\}$ such that $u_n(x)+v_n(-x)\subseteq V_n$, and therefore $-x+u_n(x)\subseteq V_n$, i.e., $u_n(x)\subseteq x+V_n$. Q.E.D.

The three conditions above are not only necessary, but sufficient for a linear space which is also a ranked space to be a linear ranked space. In other words, (I), (II) follow from (RL_1), (RL_2), (RL_3). It is clear that (RL_2) (iii) can be omitted if every V in $\mathfrak{V}$ is circled.

Proof. (I) Let $\{u_n(x)\}$, $\{v_n(y)\}$ be any fundamental sequences. From (RL_3), there are $\{U_n\}$, $\{V_n\}$ such that $u_n(x)\subseteq x+U_n$, $v_n(y)\subseteq y+V_n$. Applying (RL_1), there is a $\{W_n\}$ such that $U_n+V_n\subseteq W_n$. From (RL_3) again, there is a $\{w_n(x+y)\}$ such that $x+y+W_n\subseteq w_n(x+y)$. Thus, $u_n(x)+v_n(y)\subseteq(x+U_n)+(y+V_n)\subseteq x+y+W_n\subseteq w_n(x+y)$.

(II) Take any $\{u_n(x)\}$ and $\{\lambda_n\}$ with $\lim\lambda_n=\lambda$. From (RL_3), there is a $\{U_n\}$ such that $u_n(x)\subseteq x+U_n$. Putting $\mu_n=\lambda_n-\lambda$, we have $\lim\mu_n=0$, and therefore by (RL_2) (i), (ii), (iii), respectively, there are $\{V_n^1\}$, $\{V_n^2\}$, $\{V_n^3\}$ such that $\lambda U_n\subseteq V_n^1$, $\mu_n x\in V_n^2$, $\mu_n U_n\subseteq V_n^3$. Then, $\lambda_n u_n(x)\subseteq(\lambda+\mu_n)(x+U_n)\subseteq\lambda x+\lambda U_n+\mu_n x+\mu_n U_n\subseteq\lambda x+V_n^1+V_n^2+V_n^3$. Applying ($RL_1$), there is a $\{V_n\}$ such that $V_n^1+V_n^2+V_n^3\subseteq V_n$. Finally, from ($RL_3$) again, there is a $\{v_n(\lambda x)\}$ such that $\lambda x+V_n\subseteq v_n(\lambda x)$. Thus, we have a $\{v_n(\lambda x)\}$ such that $\lambda_n u_n(x)\subseteq v_n(\lambda x)$. Q.E.D.

In many important examples it seems natural to take $\{x+V;\ V\in\mathfrak{V}_n\}$ as $\mathfrak{V}_n(x)$. If we do so, (RL_3) is automatically fulfilled. Thus, when in a linear space E, families $\mathfrak{V}_n$ are given and satisfy axioms (A), (B), (a), (b), (RL_1), RL_2), E becomes a linear ranked space taking $\mathfrak{V}_n(x)$ as above.

It is easily seen that axioms (1), (2), (3) in [2] are sufficient conditions for (RL_1) (RL_2) (i), (ii), respectively, when every V in $\mathfrak{V}$ is circled.

§3. Examples. As remarked above, all examples in [2] are linear ranked spaces. We shall give an example which is not a linear ranked space in earlier sense.

Let Φ be the union space of countably normed spaces $\Phi^{(p)}(p=1, 2, \cdots)$ [5], i.e. $\Phi=\bigcup_{p=1}^{\infty}\Phi^{(p)}$, where

(1) $\Phi^{(p)}\subsetneqq\Phi^{(p+1)}$

(2) the systems $\{\|\ \ \|_n^{(p)}\}_{n=1,2,\ldots}$[1] and $\{\|\ \ \|_n^{(p+1)}\}_{n=1,2,\ldots}$ are equivalent in $\Phi^{(p)}$.

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1) $\{\|\ \ \|_n{}^{(p)}\}_{n=1,2,\cdots}$ will denote the system of norms in $\Phi^{(p)}$. We assume that $\|\ \ \|_1{}^{(p)}\leq\|\ \ \|_2{}^{(p)}\leq\cdots$.

Put $v(n,p\,;0)=\left\{\varphi\in\Phi^{(p)}\,;\,\|\varphi\|_n^{(p)}<\dfrac{1}{n}\right\}$, $\mathfrak{V}_n=\{v(n,p\,;0)\,;\,p=1,2,\cdots\}$ for $n\geq 1$, and $\mathfrak{V}_0=\{\Phi\}$. It is clear that, if $m\geq n$, then $v(m,p\,;0)\subseteq v(n,p\,;0)$. Moreover it can be shown that, if $v(m,p\,;0)\subseteq v(n,q\,;0)$, then necessarily $p\leq q$.

Obviously axioms (A), (a), (b) are satisfied. To prove (B), we take any $U=v(m,p\,;0)$ and $V=v(n,q\,;0)$. We may assume $p\leq q$. From the hypothesis (2), there are m' and M such that $\|\varphi\|_n^{(q)}\leq M\|\varphi\|_{m'}^{(p)}$ for $\varphi\in\Phi^{(p)}$. Taking sufficient large m'', we have $m''\geq m$ and $\|\varphi\|_n^{(q)}\leq\dfrac{m''}{n}\|\varphi\|_{m''}^{(p)}$ for $\varphi\in\Phi^{(p)}$. Now, $W=v(m'',p\,;0)\subseteq U\cap V$.

Thus, taking $\mathfrak{V}_n(\varphi)=\{\varphi+V\,;\,V\in\mathfrak{V}_n\}$, Φ becomes a ranked space. We shall show that convergence of sequences in Φ is equivalent to usual one; we have $\{\lim\varphi_i\}\ni 0$ if and only if all φ_i belong to $\Phi^{(p)}$ for some fixed p, and $\{\varphi_i\}$ converges to 0 in $\Phi^{(p)}$, i.e. $\lim\limits_i\|\varphi_i\|_n^{(p)}=0$ for each n. If $\{\lim\varphi_i\}\ni 0$, there is a $\{U_i\}$ such that $\varphi_i\in U_i$. Let $U_i=v(n_i,p_i\,;0)$. Since $U_i\supseteq U_{i+1}$, we have $p_i\geq p_{i+1}$, and therefore, for some i_0, $p_i=p\left(=\min\limits_i.\,p_i\right)$ when $i\geq i_0$. Since $n_i\uparrow\infty$, we have $\|\varphi_i\|_n^{(p)}\to 0$ for each n. Thus φ_i belongs to $\Phi^{(p)}$ for $i\geq i_0$ and converges to 0 in $\Phi^{(p)}$. From the hypothesis (2), $\{\varphi_i\}_{i\geq i_0}$ converges to 0 in $\Phi^{(p_1)}$, too. Obviously all φ_i belong to $\Phi^{(p_1)}$, and converges to 0 in $\Phi^{(p_1)}$.

On the other hand, if for some fixed p, $\varphi_i\in\Phi^{(p)}$ and φ_i converges to 0 in $\Phi^{(p)}$, we can choose an increasing sequence of positive integers, $\{i_n\}$, such that $\|\varphi_i\|_n^{(p)}<\dfrac{1}{n}$ for $i\geq i_n$. Putting $U_i=v(n,p\,;0)$ for i with $i_n\leq i<i_{n+1}$, and $U_i=\Phi$ for $i<i_1$, we get a fundamental sequence $\{U_i\}$ such that $\varphi_i\in U_i$.

Now, we shall prove (RL_1). Let $\{U_i\}$, $\{V_i\}$ be fundamental sequences, where $U_i=v(l_i,p_i\,;0)$, $V_i=v(m_i,q_i\,;0)$. We must make a $\{W_i\}$ such that $U_i+V_i\subseteq W_i$. As shown before, there are p, q, N such that $p_i=p$ and $q_i=q$ for $i\geq N$. We can assume $N=1$. If $p=q$, putting $W_i=v(n_i,p\,;0)$, where $n_i=\min.\left(\left[\dfrac{l_i}{2}\right],\left[\dfrac{m_i}{2}\right]\right)$, we have $U_i+V_i\subseteq W_i$. When $p\neq q$, we may suppose $p>q$. Since systems $\{\|\ \ \|_n^{(p)}\}$ and $\{\|\ \ \|_n^{(q)}\}$ are equivalent in $\Phi^{(q)}$ and $m_i\uparrow\infty$ there exist i_1 and C such that $\dfrac{m_i}{2}\|\varphi\|_{m_i}^{(q)}\geq\|\varphi\|_1^{(p)}$ for any $\varphi\in\Phi^{(q)}$, and for $i\geq i_1$. Thus, when $i\geq i_1$, $\|\varphi\|_1^{(p)}<\dfrac{1}{2}$ for any $\varphi\in V_i$. Repeating this process, we obtain an increasing sequence $\{i_\nu\}$ such that, when $i\geq i_\nu$, $\|\varphi\|_\nu^{(p)}<\dfrac{1}{2\nu}$ for $\varphi\in V_i$. Moreover, we can assume $l_i\geq 2\nu$ for $i\geq i_\nu$. Now, putting $W_i=v(\nu,p\,;0)$ for i with

$i_\nu \leq i < i_{\nu+1}$, and $W_i = \Phi$ for $i < i_1$, we have $U_i + V_i \subseteq W_i$.

Clearly every V in $\mathfrak{V}$ is circled. The axioms (2) and (3) in [2] hold, putting $\psi(\lambda, \mu) = \left[\frac{\lambda}{\mu}\right]$. Hence ($RL_2$) is fulfilled.

Finally we remark that Φ may not satisfy the axiom (1) in [2]; suppose that the inequalities $\|\varphi\|_1^{(p)} \geq p \cdot \|\varphi\|_p^{(1)}$ $(p=1, 2, \cdots)$ hold for every $\varphi \in \Phi^{(1)}$.[2] Then, for any l and m there are U and V, respectively in $\mathfrak{V}_l$ and $\mathfrak{V}_m$ such that no W in $\mathfrak{V}_n$, $n \geq 1$, can include $U+V$. In fact, let $U = v(l, 1; 0)$, $V = v(m, l+1; 0)$, and suppose $U + V \subseteq W$, where $W = v(n, p; 0)$, $n \geq 1$. Since $V \subseteq W$, $p \geq l+1$. For any $\varphi \in W$, $\|\varphi\|_n^{(p)} < \frac{1}{n}$, a fortiori $\|\varphi\|_1^{(p)} < 1$. If we take a $\varphi \in \Phi^{(1)}$ such that $\|\varphi\|_1^{(p)} = 1$, clearly $\varphi \notin W$. On the other hand, $1 = \|\varphi\|_1^{(p)} \geq p\|\varphi\|_p^{(1)} \geq (l+1)\|\varphi\|_{l+1}^{(1)} \geq (l+1)\|\varphi\|_l^{(1)}$, therefore $\|\varphi\|_l^{(1)} \leq \frac{1}{l+1} < \frac{1}{l}$, i.e. $\varphi \in U$. Hence $U \not\subseteq W$. This contradicts the fact that $U + V \subseteq W$.

§4. Bounded sets in linear ranked spaces. We already gave a definition of bounded sets in linear ranked spaces in earlier sense in [3], and another definition in [4]. Now, let E be a linear ranked space in new sense. We use Definition 2 in [4]: A subset B in E is called bounded if there is a fundamental sequence $\{V_n\}$, any member of which absorbs B. The study for bounded sets in [4] can be applied to our case. For example, from (RL_1), it follows that any finite union and finite sum of bounded sets are also bounded, from (RL_2) (i), that any scalar multiple of bounded set is bounded, and from (RL_3) (ii), that any one point set is bounded.

We give a sufficient condition for the property that every convergent sequence is bounded. It is as follows: For any $\{U_n\}$ there is a $\{V_n\}$ such that every V_n is circled and for some N, $\bigcup_\lambda \lambda V_n = \bigcup_\lambda \lambda V_N$ for $n \geq N$. The proof is trivial and omitted. The union of countably normed spaces Φ in §3 evidently satisfies this condition.

Now, we shall show that, in Φ, boundedness is equivalent to usual one; B is bounded in our sense, if and only if B is included in some $\Phi^{(p)}$ and $\sup_{\varphi \in B} \|\varphi\|_n^{(p)} < \infty$ for each n. In fact, suppose that, for some $\{V_i\}$, where $V_i = v(n_i, p_i; 0)$, every V_i absorbs B, and let $p = \min. p_i$. Clearly, $B \subset \Phi^{(p)}$ and $\sup_{\varphi \in B} \|\varphi\|_n^{(p)} < \infty$ for each n. On the other hand if $B \subset \Phi^{(p)}$ and $\sup_{\varphi \in B} \|\varphi\|_n^{(p)} < \infty$ for each n, then putting $U_n = v(n, p; 0)$, we get a $\{U_n\}$ any member of which absorbs B.

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2) This is possible, when we omit some finite members of $\{\| \ \|_n^{(p)}\}$, and multiply by some positive numbers. In each $\Phi^{(p)}$, the new system of norms is equivalent to initial one, and therefore convergence of sequence in Φ is unaltered.

Finally we remark that first definition of boundedness in [3] (Definition 1 in [4]) is not always equivalent to usual one. To prove this, we put $\Phi=\mathcal{D}$, $\Phi^{(p)}=\mathcal{D}_p=\{\varphi\in\mathcal{D}\,;\,\mathrm{car}\,\varphi\subseteq[-p,p]\}$ and define the systems of norms as follows. Let $\|\varphi\|_n=\max_{0\le j\le n-1}\sup_x|\varphi^{(j)}(x)|$. In $\Phi^{(p)}$, let $|\varphi|_1^{(p)}=\sup_{\|\psi\|_p\le 1}|\varphi(\psi)|$[3], $\cdots$ $|\varphi|_p^{(p)}=\sup_{\|\psi\|_1\le 1}|\varphi(\psi)|$, $|\varphi|_{p+n}^{(p)}=\|\varphi\|_n (n=1,2,\cdots)$. Obviously two systems $\{\|\ \ \|_n\}$ and $\{|\ \ |_n^{(p)}\}$ are equivalent in $\Phi^{(p)}$, and therefore convergence of sequences in Φ coincides with usual one. In this space Φ, $V=v(2,1\,;0)=\left\{\varphi\in\mathcal{D}_1\,;\,|\varphi|_2^{(1)}=\sup_x|\varphi(x)|<\frac{1}{2}\right\}$ is bounded by Definition 1; for any n there is a U in $\mathfrak{V}_n$ which absorbs V. In fact, let $U=v(n,n-1\,;0)$. Since $|\varphi|_n^{(n-1)}=\|\varphi\|_1$, $U=\left\{\varphi\in\mathcal{D}_{n-1}\,;\,\sup_x|\varphi(x)|<\frac{1}{n}\right\}$. Hence $\frac{2}{n}V\subseteq U$. It is clear that this set is not bounded in usual sense.

References

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3) $\varphi(\psi)=\int_{-\infty}^{\infty}\varphi(x)\psi(x)dx$ where $\varphi,\psi\in\mathcal{D}$.