82. *On Classes of Summing Operators. I*

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(Comm. by Kinjirô KUNUGI, M. J. A., May 12, 1969)

1. Let X and Y be Banach spaces and $L(X, Y)$ be the space of all bounded operators from X into Y with the norm

$$\|T\| = \sup_{\|x\|\le 1} \|Tx\|$$

Some important classes of operators in $L(X, Y)$ were considered in connection with the classes of nuclear operators and Hilbert-Schmidt operators; a unified theory of these is the theory of p-absolutely summing operators of A. Pietsch [4].

It is the aim of the present note to give a generalization of the class of A. Pietsch. In the first step our generalization has its origin in the generalization of L_p-spaces given by Orlicz.

In a paper which will follow we give a new generalization inspired from theory of modulared spaces [3].

2. Complementary functions in the sense of Young.

For $t \geqslant 0$ let $y = \varphi(t)$ be a non-decreasing function such that $\varphi(0) = 0$, φ does not vanish identically and φ is left continuous for $t > 0$; let ψ be the left continuous inverse of φ. Define the function

$$\phi(t) = \int_0^t \varphi(s)ds \qquad \psi(s) = \int_0^s \psi(r)dr$$

for $t, s \geqslant 0$. The pair (ϕ, ψ) is called complementary Young functions; a basic result is

$$ts \leqslant \phi(t) + \psi(s)$$

and equality holds if and only if $s = \varphi(t)$, $t = \psi(s)$.

3. ϕ-absolutely summing operators.

Let X, Y be Banach spaces and $T \in L(X, Y)$.

Definition 1. For $T \in L(X, Y)$ and (ϕ, ψ) be complementary Young functions we define the number $a_\phi(T)$ us

$$a_\phi(T) = \inf\left\{c, \sum_1^n \phi(\|Tx_i\|)\right\} \leqslant \phi(c) \sup_{\|x^*\|\le 1} \left(\sum_1^n \phi(|x^*(x_i)|)\right)$$
$$x_i \in X,\ i = 1, 2, \cdots, n.$$

If $a_\phi(T) < \infty$ we call T, ϕ-absolutely summing.

Remark. Since $\phi(t) = \dfrac{1}{p} t^p$, $\psi(s) = \dfrac{1}{q} s^q$ is a pair of complementary functions, for this functions we obtain the class of A. Pietsch.

In what follows, we suppose that ϕ is with Δ_2^1-property with $u_0=0$ (which implies the Δ_2-property with $u_0=0$ [1]).

4. Properties of ϕ-absolutely summing operators.

We note the set of all ϕ-absolutely summing operators by $L_\phi(X, Y)=L_\phi$. Our result is

Theorem 1. *L_ϕ is a linear space.*

Proof. If $T, S \in L_\phi$ then

$$\begin{aligned}\sum_1^n \phi\|(T+S)x_i\| &\leqslant \sum_1^n \phi(\|Tx_i\|+\|Sx_i\|)\\ &=\sum_1^n \phi\left(\frac{1}{2}(2\|Tx_i\|+2\|Sx_i\|)\right)\leqslant \frac{1}{2}\sum_1^n(\phi(2\|Tx_i\|)+\phi(2\|Sx_i\|)\\ &\leqslant \frac{k}{2}\sum_1^n \phi\|Tx_i\|)+\phi\ (\|Sx_i\|)\end{aligned}$$

which shows that $(T+S)\in L_\phi$. Also,

$$\begin{aligned}\sum_1^n \phi(\|\alpha Tx_i\|)&=\sum_1^n \phi(|\alpha|\,\|Tx_i\|)\\ &\leqslant \sum_1^n \phi(2^m\|Tx_i\|)\leqslant k^m \sum \phi(\|Tx_i\|)\end{aligned}$$

where $\quad 2^m \geqslant |\alpha|$

and the theorem is proved.

The following theorem gives a characterization of ϕ-absolutely summing operators in terms of integral representations,

Theorem 2. *$T\in L(X, Y)$ is ϕ-absolutely summing if and only if there exists a Radon measure μ on the unit ball of X^*, U such that*

$$\phi(\|Tx\|)\leqslant \phi(a_{\phi(T)})\int_U \phi\,|\varphi_x(a)|\,d\mu(a)$$

(here $\varphi_x(a)=\langle x, a\rangle$ the values of the functional a in the point x).

Proof. The proof of this result is modeled on the proof for the case $\phi(t)=\frac{1}{p}t^p$ considering the functional

$$s(\varphi)=\inf_{x_1,\cdots,x_n}\left\{\sup_{\|a\|\le 1}\left\{\varphi(a)+\phi(c)\sum_1^n \phi\,|\langle x_i, a\rangle| - \sum_1^n \phi(\|Tx_i\|)\right\}\right\}$$

where $\varphi\in C(U)=$ the space of all real continuous functions on U. The properties of $s(\varphi)$ are the same as in the case $\phi(t)=\frac{1}{p}t^p$. Applying Hahn-Banach theorem and Mazur-Orlicz theorem we find that there exists a Radon measure μ such that

$$\langle \varphi, \mu\rangle \leqslant s(\varphi).$$

Also μ is positive and since

$$\langle 1, \mu\rangle \leqslant s(1)=1$$

it follows that μ is a probability measure.

It is clear that

$$\phi(\|Tx\|)\leqslant\phi(c)\int_U\phi(|\varphi_x(a)|)d\mu(a)$$
$$=\phi(c)\int_U\phi(|\langle x,a\rangle|)d\mu$$

since $\varphi_x(a)=\langle x,a\rangle$ is in $C(U)$. The theorem is proved. From this theorem it is easy to produce a new proof of Theorem 1.

References

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