78. Generalizations of M-spaces. I

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(Comm. by Kinjirô KUNUGI, M. J. A., May 12, 1969)

In this paper we shall give some generalizations for the notion of M-spaces introduced by K. Morita [8]. A space X is called an *M-space* if there exists a normal sequence $\{\mathfrak{U}_i\}$ of open coverings of X satisfying the following condition (M) below:

(M) If $\{K_i\}$ is a decreasing sequence of non-empty closed sets of X such that $K_i \subset \mathrm{St}(x_0, \mathfrak{U}_i)$ for each i and for a fixed point x_0 of X, then $\cap K_i \neq \phi$.

From condition (M) we obtain further a *condition* (M') (resp. (M_δ)) with the phrase "K_i is a closed set" replaced by "K_i is a zero set" (resp. "K_i is a closed G_δ-set") and we shall call a space X an *M'-space* (resp. *M_δ-space*) if X satisfies the condition (M′) (resp. (M_δ)). The class of M'-spaces contains all pseudocompact spaces and all M-spaces. There are properties for M'-spaces similar to those for M-spaces, for instance, an M'-space X has Morita's paracompactification μX which is obtained by K. Morita for M-spaces. Moreover, as a nice property of M'-space, any subspace of μX, containing X, is always an M'-space while this property does not hold in case X is an M-space.

For simplicity, we assume that all spaces are completely regular T_1-spaces and that mappings are continuous; we denote by βX and υX the Stone-Čech compactification and Hewitt realcompactification of a given space X respectively. For a mapping $\varphi: X \to Y$, the symbol Φ denotes the Stone extension of φ from βX onto βY. N is the set of all natural numbers. Other terminologies and notations will be used as in [3].

1. Characterization of M'-spaces.

Let φ be a mapping from X onto Y. φ is a *WZ-mapping* if $\mathrm{cl}_{\beta X}\varphi^{-1}(y) = \Phi^{-1}(y)$ for each $y \in Y$ [7] and φ is a Z (resp. Z_δ)-*mapping* if $\varphi(F)$ is closed for each zero set (resp. closed G_δ-set) F of X. A Z (resp. Z_δ)-mapping φ is a Z_p (resp. $Z_{\delta p}$)-*mapping* if $\varphi^{-1}(y)$ is pseudocompact for each $y \in Y$. A subset F of X is called a *relatively pseudocompact* if f is bounded on F for each $f \in C(X)$. A Z-mapping φ is said to be an *SZ-mapping* if $\varphi^{-1}(y)$ is relatively pseudocompact for each $y \in Y$.

K. Morita [8] has proved that X is an M-space if and only if there exists a quasi-perfect mapping φ from X onto some metric space Y where a closed mapping φ is called a *quasi-perfect* mapping if $\varphi^{-1}(y)$

is countably compact for each $y \in Y$. The proof of the following theorem is a modification of K. Morita's and hence we shall only state in different points.

Theorem 1.1. *A space X is an M'-space (resp. M_δ-space) if and only if there exists an SZ (resp. $Z_{\delta p}$)-mapping from X onto some metric space Y.*

Proof. Since the "if" part is the very same as one of Theorem 6.1 in [8], we shall prove only the "only if" part. Let $(X, \mathfrak{U})$ be a space obtained from X by taking $\{\mathrm{St}(x, \mathfrak{U}_i); i \in N\}$ as a basis of neighborhoods at each point x of X and φ_1 the identity mapping of X onto $(X, \mathfrak{U})$. We introduce a relation "$\sim$" in $(X, \mathfrak{U})$ defining by "$x \sim y$" if $y \in \cap \mathrm{St}(x, \mathfrak{U}_i)$ and denote by Y the quotient space obtained from this relation and φ_2 the quotient mapping from $(X, \mathfrak{U})$ onto Y. It is obvious that Y is metrizable and $\varphi = \varphi_2\varphi_1$ is continuous. Suppose that $A = Z(f)$ is a zero set of X and $y_0 \in \overline{\varphi(A)}$ and $x_0 \in \varphi^{-1}(y_0)$. Since φ_2 is known to be open,

$$B_i = \varphi_2(\mathrm{int}\{x\,;\, \mathrm{St}(x, \mathfrak{U}_n) \subset \mathrm{St}(x_0, \mathfrak{U}_i) \text{ for some } n\})$$

is open and contains y_0. From this we have $\mathrm{St}(x_0, \mathfrak{U}_i) \cap A \neq \phi$ $(i \in N)$. Let d be a distance function on Y. B_i being open in Y, there is a positive number r_i such that $\{r_i\} \downarrow 0$ and

$$F_i = \{y\,;\, d(y_0, y) \leqq r_i\} \subset B_i \text{ and } \mathrm{int}F_i \cap \varphi(A) \neq \phi.$$

Then $F_i = Z(g_i)$ where $g_i(y) = d(y_0, y) \vee r_i - r_i$, and $E_i = \varphi^{-1}F_i = Z(g_i\varphi)$ is a zero set of X and $Z_i = E_i \cap A (\neq \phi)$ is also a zero set of X. By the condition (M$'$) we have $\cap Z_i \neq \phi$. If $x_1 \in \cap Z_i$, then $x_1 \in \cap \mathrm{St}(x_0, \mathfrak{U}_i)$ which shows that $\varphi(A)$ is closed.

Next we shall prove that $\varphi^{-1}(y)$ is relatively pseudocompact for each $y \in Y$. If there exists a positive function $f \in C(X)$ which is unbounded on $\varphi^{-1}(y)$, then $Z_n = \{x\,;\, f(x) \geqq n\} \cap \varphi^{-1}(y)$ is a zero set of X because $\varphi^{-1}(y)$ is a zero set of X, and $\{Z_n\}$ is decreasing. Since $Z_n \subset \mathrm{St}(x_0, \mathfrak{U}_n)$ $(n \in N)$ and for a fixed point x_0 in $\varphi^{-1}(y)$, the condition (M$'$) implies that $\cap Z_n \neq \phi$ which is a contradiction. The proof for an M_δ-space is the very same as one of an M'-space.

Remark 1.2. A space X is said to be an *M_{zp}-space* if there exists a Z_p-mapping from X onto some metric space Y. It is easy to see that the following implications hold:

$$M\text{-spaces} \rightarrow M_\delta\text{-spaces} \rightarrow M_{zp}\text{-space} \rightarrow M'\text{-space}$$

and that if X is normal, then these four spaces coincide (cf. [7], 1.3).

Corollary 1.3. *Every pseudocompact space is an M_δ-space.*

In the next paper it is shown that a non-countably compact, pseudocompact space is not an M-space. Since a mapping from a pseudocompact space onto a metric space is always an SZ-mapping ([7], 1.5 and Theorem 2.1), a product of a pseudocompact space with a metric space is an M_{zp}-space.

2. Some properties of M'-spaces.

A space X is said to be *topologically complete* if there is a uniformity for X relative to which X is complete. The following lemmas will be used in this section.

Lemma 2.1. *If F is a relatively pseudocompact subset of a subspace of Z and F is dense in $E\subset Z$, then E is a relatively pseudocompact subset of Z.*

Lemma 2.2. *If F is a relatively pseudocompact closed subset of a topologically complete space, then F is compact* (cf. [2]).

Lemma 2.3. *If φ is a perfect mapping from X onto Y, then X is a paracompact M-space if and only if so is Y* ([4], [6], [9]).

Lemma 2.4. *If φ is a WZ-mapping from X onto a metric space Y such that $\varphi^{-1}(y)$ is relatively pseudocompact for each $y\in Y$, then φ is a Z-mapping and hence φ is an SZ-mapping* ([7], 1.4 and 3.1).

If X is an M'-space, then there exists some metrizable space mentioned in Theorem 1.1. But such a metric space is not necessarily unique and hence we shall denote by $M(X)$ the set of all such metrizable spaces and we set $\mu_Y(X)=\Phi^{-1}(Y)$ ($Y\in M(X)$). $\Phi\,|\,\mu_Y(X)$ is obviously a perfect mapping from $\mu_Y(X)$ onto Y. Since Y is a metric space, Y is a paracompact M-space and by Lemma 2.3 $\mu_Y(X)$ is a paracompact M-space.

Theorem 2.5. *If X is an M'-space and φ is an SZ-mapping from X onto a metrizable space Y, then, in βX, $\mu_Y(X)$ is the smallest topologically complete subspace containing X.*

Proof. Suppose that $X\subset W\subset\beta X$ and W is topologically complete. φ being a SZ-mapping, $\mathrm{cl}_{\beta X}\varphi^{-1}(y)=\Phi^{-1}(y)$ for each $y\in Y$. $\varphi^{-1}(y)$ is relatively pseudocompact in X and dense in a closed subset $W\cap\Phi^{-1}(y)$ of W, and hence $W\cap\Phi^{-1}(y)$ is relatively pseudocompact in W by Lemma 2.1. Since W is topologically complete, $\Phi^{-1}(y)\cap W$ is compact by Lemma 2.2. This leads that $\Phi^{-1}(y)=\Phi^{-1}(y)\cap W$, i.e., $\mu_Y(X)\subset W$.

Remark 2.6. This theorem means that $\mu_Y(Y)=\mu_Z(Z)$ for all $Y,Z\in M(X)$ and hence we denote by μX, called a *Morita's paracompactification of X*, the paracompact M-space determined uniquely in the sense above. This theorem for M-spaces has been obtained by K. Morita [10].

Corollary 2.7. *If an M'-space X is topologically complete, then X is a paracompact M-space. Particularly, if X is a realcompact M'-space, then X is a paracompact M-space.*

As is shown in the next paper, there is an M-space X such that some subspace, of μX, containing X is not an M-space. But we have the following theorem for M'-spaces.

Theorem 2.8. *If X is an M'-space, then every subspace W of μX such that $X \subset W \subset \mu X$ is always an M'-space.*

Proof. Let φ be an SZ-mapping from X onto a metric space Y and $\varphi_1 = \Phi | W$. Then $\varphi_1^{-1}(y)$ is relatively pseudocompact in W for every $y \in Y$ by Lemma 2.1 and Theorem 2.5. From Lemma 2.4 φ_1 is a Z-mapping. Thus φ is an SZ-mapping from W onto Y which shows that W is an M'-space.

Now suppose that there is a realcompact space $Y \in M(X)$. By Theorem 2.5, $\mu X \subset \upsilon X$ because υX is topologically complete. On the other hand, μX is a preimage of a realcompact space Y under a perfect mapping and hence μX is realcompact ([3] or [7]). υX being the smallest realcompact space of βX containing X, we have $\upsilon X \subset \mu X$ which shows that $\mu X = \upsilon X$. For any $Z \in M(X)$, Z is an image of a realcompact M-space under a perfect mapping and Z is realcompact ([5] or [7]). From these we have

Theorem 2.9. *Let X be an M'-space, then*

1) *if there exists a realcompact space in $M(X)$, then so is every space in $M(X)$ and $\mu X = \upsilon X$ and μX is a paracompact realcompact M-space,*

2) *if there exists a non-realcompact space in $M(X)$, then so is every space in $M(X)$ and $\mu X \subsetneqq \upsilon X$ and μX is a paracompact M-space which is not realcompact.*

Similarly to Theorem 2.9 we have

Theorem 2.10. *Let X be an M'-space. If there exists a space $Y \in M(X)$ which is topologically complete in the sense of Čech, then so is every space in $M(X)$ and μX is a paracompact M-space which is topologically complete in the sense of Čech.*

A space X is *locally pseudocompact* if every point of X has a pseudocompact neighborhood. As in [1], we have

Theorem 2.11. *If X is an M'-space, then X is locally pseudocompact if and only if there exists a locally compact space Y such that $X \subset Y \subset \mu X$.*

Theorem 2.12. *If X is an M'-space, then the followings are equivalent:*

1) *X is locally compact.*

2) *Every space in $M(X)$ is locally compact.*

3) *There exists a locally compact space in $M(X)$.*

4) *There exists a space $Y \in M(X)$ such that $\varphi^{-1}(y)$ is contained in a pseudocompact neighborhood for each $y \in Y$ where φ is an SZ-mapping from X onto Y.*

5) *For each $p \in \mu X$, there exist pseudocompact subsets A and B*

of X such that $p \in \mathrm{cl}_{\mu X} A$, $f=0$ on A and $f=1$ on B^c for some $f \in C(X)$.

Proof. 5)$\leftrightarrow$1)$\rightarrow$4) follows essentially as in [1]. 1)$\leftrightarrow$2)$\leftrightarrow$3) are similar to the proof of Theorem 2.9. 4)$\rightarrow$2) is obtained from the fact that $\mathrm{cl}_{\mu X} V$ is a compact neighborhood of $\Phi^{-1}(y)$ where V is a pseudocompact neighborhood of $\varphi^{-1}(y)$.

A subset F of X is said to be *Z-embedded in X* if for every zero set Z of F, there exists a zero set Z' of X such that $Z=Z' \cap F$. If F is Z-embedded and completely separated from any zero sets disjoint from it, then F is C-embedded (cf. [2]). Thus a zero set is C-embedded if and only if it is Z-embedded. Since a Z-embeddable pseudocompact subset is pseudocompact, we have

Theorem 2.13. *Let φ be an SZ-mapping from X onto a metric space Y, then $\varphi^{-1}(y)$ is Z-embedded for each $y \in Y$ if and only if $\Phi^{-1}(y) = \upsilon\varphi^{-1}(y)$ for every $y \in Y$ (in this case X is an M_{zp}-space).*

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