76. *A Class of Purely Discontinuous Markov Processes with Interactions.* I[1]

By Tadashi UENO
Department of Mathematics, Faculty of General Education, University of Tokyo

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1. Starting with Kac's model of Boltzmann equation,[2] McKean [5]-[7] introduced an interesting class of Markov processes with nonlinear generators. These processes describe the motion of one particle under the interactions between infinite number of similar particles.[3]

We construct a class of these processes by modifying the classical method of Feller [1]. The forward equation with possibly unbounded and temporally inhomogeneous equation is considered. Interactions can be infinitely multifold.

I thank S. Tanaka and H. Tanaka who sent me the manuscript of [9] and a part of [8], respectively, before publication.

2. First, we consider the simpliest model with binary interactions. Let R be a locally compact space with countable bases and let $B(R)$ be the topological Borel field. The forward equation is

$$(1)\qquad \frac{d}{dt}P^{(f)}(s,x,t,E)=\int_R P^{(f)}(s,x,t,dy)A^{(P^{(f)}_{s,t})}(t,y,E),$$
$$-\infty\leq t_0\leq s<t\leq t_1\leq+\infty$$
$$P^{(f)}(s,x,t,E)\to\delta_x(E),\quad \text{as } t\to s,$$

where initial distribution f at time s and the solution $P^{(f)}(s,x,t,E)$ are substochastic measures, and

$$P^{(f)}_{s,t}(E)=\int_R f(dx)P^{(f)}(s,x,t,E).$$

Kernel $A^{(u)}$ indexed by a substochastic measure u, is

$$(2)\qquad A^{(u)}(t,x,E)=\int_R u(dx_1)q(x_1\,|\,t,x)(\pi^0(x_1\,|\,t,x,E)-\delta_x(E)),$$

where $q(x_1\,|\,t,x)$ is non-negative and majorized by another function $q(t,x)$ which is *bounded on any compact* (t,x)*-set.* $\pi^0(x_1\,|\,t,x,E)$ is a *probability measure with no mass at point* x. $q(t,x)$, $q(x_1\,|\,t,x)$ and $\pi^0(x_1\,|\,t,x,E)$ are measurable in (t,x) and (x_1,t,x), and continuous in t when other variables are fixed. Intuitively, $\pi^0(x_1\,|\,t,x,E)$ indicates the

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1) Research supported by the N S F at Cornell University.

2) Introduced by Kac [4] related with a justification of Boltzmann equation.

3) This explanation is justified by the "propagation of chaos" proposed by Kac. The reader can consult Kac [4] and McKean [5, 6].

hitting measure to $R-\{x\}$ from point x at time t under the influence of another particle at point x_1. Similarly, $q(x_1|t, x)$ determines the waiting time until jump depending on the value of x_1. If π^0 and q are independent on x_1, and hence u of $A^{(u)}$ is ignored, then $A(t, x, E)$ reduces to an ordinary generator of a purely discontinuous Markov process in [1].

We rewrite (2) using $\pi(x_1|t, x, E)$ below:

$$(2') \qquad A^{(u)}(t, x, E)=q(t, x)\int_R u(dx_1)(\pi(x_1|t, x, E)-\delta_x(E)),$$
$$\pi(x_1|t, x, E)=q(t, x)^{-1}\{q(x_1|t, x)\pi^0(x_1|t, x, E)+(q(t, x)-q(x_1|t, x))\delta_x(E)\}$$

By a solution of (1), we mean a substochastic measure $P^{(f)}(s, x, t, E)$, absolutely continuous in t, for which the right side of (1) is finite and (1) holds excepting a t-set of Lebesque measure 0. And we consider (1) only for *bounded set* E.[4] This is equivalent with

$$(1') \quad P^{(f)}(s, x, t, E)-\delta_x(E)=\int_s^t dr\int_R P^{(f)}(s, x, r, dy)q(r, y)\int_R P_{r,s}^{(f)}(dx_1)(\pi(x_1|t, y, E)-\delta_y(E)),$$
$$E \text{ bounded}.$$

Our main results about the forward equation are formulated in terms of

$$(3) \quad P^{(f)}(s, x, t, E)=e^{-\int_s^t q(\sigma, x)d\sigma}\delta_x(E)+\int_s^t dr\int_R P^{(f)}(s, x, r, dy)q(r, y)\int_R P_{s,r}^{(f)}(dx_1)\int_E \pi(x_1|r, y, dz)e^{-\int_r^t q(\sigma, z)d\sigma}.$$

Theorem. i) *For each initial distribution* f, (3) *has a substochastic solution* $p^{(f)}(s, x, t, E)$ *which is majorized by any substochastic solution of* (3). ii) *This minimal solution of* (3) *satisfies a version of Chapman-Kolmogorov equation*:

$$(4) \quad P^{(f)}(s, x, u, E)=\int_R P^{(f)}(s, x, t, dy)P^{(P_{s,t}^{(f)})}(t, y, u, E), s<t<u.$$

iii) (1′) *has a stochastic solution if and only if* (3) *has a stochastic solution. A properly substochastic solution*[5] *of* (3) *never satisfies* (1′), *and conversely. In particular, if the minimal solution of* (3) *is stochastic, then the minimal solution is the unique solution of* (1′) *and* (3).

3. Outline of the proof. Define, inductively,

$$(5) \qquad S_0^{(f)}(s, x, t, E)=e^{-\int_s^t q(\sigma, x)d\sigma}\delta_x(E)$$

4) We call a set *bounded*, if it is contained in a compact set.

5) A substochastic kernel is called *properly substochastic*, if the total mass is less than 1.

$$S_{n+1}^{(f)}(s,x,t,E)=e^{-\int_s^t q(\sigma,x)d\sigma}\delta_x(E)$$
$$+\int_s^t dr\int_R S_n^{(f)}(s,x,r,dy)q(r,y)\int_R S_n^{(f)}(s,r,dx_1)\int_E \pi(x_1|r,y,dz)e^{-\int_r^t q(\sigma,z)d\sigma},$$
$$S_n^{(f)}(s,t,E)=\int_R f(dx)S_n^{(f)}(s,x,t,E).$$

Then, for any $E\in B(R)$,

$$(6)\quad \begin{aligned} &S_{n+1}^{(f)}(s,x,t,E)+\int_s^t dr\int_E S_{n+1}^{(f)}(s,x,r,dy)q(r,y)\\ &=\delta_x(E)+\int_s^t dr\int_R S_n^{(f)}(s,x,r,dy)q(r,y)\int_R S_n^{(f)}(s,r,dx_1)\pi(x_1|r,y,E).\end{aligned}$$

This is proved with the existence of $S_{n+1}^{(f)}$ in (5) at the same time. In fact, assume (a), (b) and (c) below for n, which are clear for $n=0$.

(a) $\int_s^t dr\int_R S_n^{(f)}(s,x,r,dy)q(r,y)<\infty$; (b) $S_0^{(f)}\leq\cdots\leq S_{n+1}^{(f)}$;

(c) $S_n^{(f)}(s,x,t,R)\leq 1$.

Here, $S_{n+1}^{(f)}(s,x,t,E)$ in (b) exists because of (a). Compute for *bounded* E,

$$\begin{aligned}\int_s^t dr\int_E S_{n+1}^{(f)}(s,x,r,dy)q(r,y)&=\int_s^t dr\int_E e^{-\int_s^r q(\sigma,x)d\sigma}\delta_x(dy)q(r,y)\\ &+\int_s^t dr\int_s^r d\tau\int_R S_n^{(f)}(s,x,\tau,dy)q(\tau,y)\int_R S_n^{(f)}(s,\tau,dx_1)\\ &\times\int_E \pi(x_1|\tau,y,dz)e^{-\int_\tau^r q(\sigma,z)d\sigma}q(r,z)\\ &=(1-e^{-\int_s^t q(\sigma,x)d\sigma})\delta_x(E)-\int_s^t d\tau\int_\tau^t dr\int_R S_n^{(f)}{}''\int_E\pi(x_1|\tau,y,dz)\frac{d}{dr}e^{-\int_\tau^r q(\sigma,z)d\sigma}\\ &=(1-e^{-\int_s^t q(\sigma,x)d\sigma})\delta_x(E)-\int_s^t d\tau\int_R S_n^{(f)}{}''\int_E\pi(x_1|\tau,y,dz)(e^{-\int_\tau^r q(\sigma,z)d\sigma}-1)\\ &=\delta_x(E)-S_{n+1}^{(f)}(s,x,t,E)+\int_s^t dr\int_R S_n^{(f)}(s,x,r,dy)q(r,y)\\ &\times\int_R S_n^{(f)}(s,r,dx_1)\pi(x_1|r,y,E),\end{aligned}$$

implying (6). Taking bounded $E_k\nearrow R$ in (6) and noting (a) for n, we have (6) for all $E\in B(R)$. Substitution $E=R$ in (6) implies (a) for $n+1$, and hence $S_{n+2}(s,x,t,R)<\infty$. Now, (b) for $n+1$ is clear by (b) for n. (c) for $n+1$ is obtained by letting $E=R$ in (6) and by noting that the second term on the right side of (6) is bounded by

$$\int_s^t dr\int_R S_n^{(f)}(s,x,r,dy)q(r,y).$$

Let $P^{(f)}(s,x,t,E)$ be the limit of $S_n^{(f)}(s,x,t,E)$. This satisfies (3) and (7) below by letting $n\to\infty$ in (5) and (6), respectively.

$$(7)\quad \begin{aligned}&P^{(f)}(s,x,t,E)-\delta_x(E)\\ &=\int_s^t dr\int_R P^{(f)}(s,x,r,dy)q(r,y)\Big(\int_R P_{s,r}^{(f)}(dx_1)\pi(x_1|r,y,E)-\delta_y(E)\Big),\end{aligned}$$

bounded E.

Since every solution of (3) majorizes $S_n^{(f)}$ inductively, $P^{(f)}$ is the minimal solution.

Chapman-Kolmogorov equation (4) is obtained by

$$(8)\quad S_n^{(f)}(s,x,u,E)\leq\int_R P^{(f)}(s,x,t,dy)P^{(P_{s,t}^{(f)})}(t,y,u,E),\qquad s<t<u$$

$$(9)\quad P^{(f)}(s,x,u,E)\geq\int_R P^{(f)}(s,x,t,dy)S_n^{(P_{s,t}^{(f)})}(t,y,u,E),\qquad s<t<u.$$

Since (8) is clear for $n=0$, we assume (8) for n. By (5),

$$S_{n+1}^{(f)}(s,x,u,E)=e^{-\int_s^t q(\sigma,x)d\sigma}e^{-\int_t^u q(\sigma,x)d\sigma}\delta_x(E)$$
$$+\int_s^t dr\int_R S_n^{(f)}(s,x,r,dy)q(r,y)\int_R S_n^{(f)}(s,r,dx_1)$$
$$\times\int_E \pi(x_1|r,y,dz)e^{-\int_r^t q(\sigma,z)d\sigma}e^{-\int_t^u q(\sigma,z)d\sigma}\delta_z(E)$$
$$+\int_t^u dr\int_R S_n^{(f)}(s,x,r,dy)q(r,y)\int_R S_n^{(f)}(s,r,dx_1)\int_E \pi(x_1|r,y,dz)e^{-\int_r^u q(\sigma,z)d\sigma}$$

By (5), the sum of the first and second terms above coincides with

$$\int_E S_{n+1}^{(f)}(s,x,t,dy)e^{-\int_t^u q(\sigma,y)d\sigma}\leq\int_E P^{(f)}(s,x,t,dy)e^{-\int_t^u q(\sigma,y)d\sigma}.$$

By (3) and the assumption (8) for n, the third term is bounded by

$$\int_R P^{(f)}(s,x,t,dy)(P^{(P_{s,t}^{(f)})}(t,y,u,E)-e^{-\int_t^u q(\sigma,y)d\sigma}\delta_y(E)),$$

implying (8) for $n+1$. (9) is proved similarly, rewriting $P^{(f)}(s,x,u,E)$ by (3).

To prove iii), note that (1′) coincides with (7) if and only if $P_{s,t}^{(f)}(\cdot)$, equivalently, $P^{(f)}(s,x,t,\cdot)$ is *stochastic*. On the other hand, (7) and (3) are equivalent. In fact, integrate $q(t,y)$ by both hand sides of (3) on a bounded set E, and then integrate from s to u as a function of t. This implies (7). Similarly, (7) implies (3), where $q(t,y)e^{-\int_t^u q(\sigma,y)d\sigma}$ is integrated instead of $q(t,y)$. The last statement about the minimal solution is clear.

4. The general case. Consider equation (1), replacing (2) by

$$(10)\quad A^{(u)}(t,x,E)=\sum_{N=0}^{\infty}\int_{R^N}\prod_{k=1}^{N}u(dx_k)q_N(x_1,\cdots,x_N|t,x)\times(\pi_N^0(x_1,\cdots,x_N|t,x,E)-\delta_x(E))$$

where $q_N(x_1,\cdots,x_N|t,x)$ are non-negative and majorized by $q_N(t,x)$, and $q(t,x)=\sum_{N=0}^{\infty}q_N(t,x)$ is bounded on any compact (t,x)-set. Measurability and continuity assumptions for q_N, q and π_N^0 are similar. π_N^0 are probability measures with no mass at point x. Then, this equation gives a model with infinitely multifold interactions.

Our main theorem in **2** holds true for this equation, where (1′) and (3) are replaced by the following equations with clear modification of π_N^0 by π_N as in (2′).

$$
(11)\quad \begin{aligned} P^{(f)}(s,x,t,E)-\delta_x(E) &= \int_s^t dr \int_R P^{(f)}(s,x,r,dy) \sum_{N=0}^{\infty} q_N(r,y) \\ &\times \int_{R^N} \prod_{k=1}^{N} P_{s,r}^{(f)}(dx_k)(\pi_N(x_1,\cdots,x_N\,|\,r,y,E)-\delta_y(E)). \end{aligned}
$$

$$
(12)\quad \begin{aligned} P^{(f)}(s,x,t,E) &= e^{-\int_s^t q(\sigma,x)d\sigma}\delta_x(E) + \int_s^t dr \int_R P^{(f)}(s,x,r,dy) \sum_{N=0}^{\infty} q_N(r,y) \\ &\times \int_{R^N} \prod_{k=1}^{N} P_{s,r}^{(f)}(dx_k) \int_E \pi_N(x_1,\cdots,x_N\,|\,r,y,dz) e^{-\int_r^t q(\sigma,z)d\sigma} \end{aligned}
$$

Each step of the proof in 3 can be applied for this equation with clear modifications.

Thus, keeping the Chapman-Kolmogorov equation in mind, we can construct an ordinary temporally inhomogeneous Markov process with initial distribution f at time s_0 and the transition probability $P(s,x,t,E) \equiv P^{(P_{s_0 s}^{(f)})}(s,x,t,E)$.[6] Properties as a stochastic process such as properties of path functions, strong Markov property, can be discussed in an ordinary way.

It is expected that the class of solutions for (3) or (1′) is explained in terms of an appropriate ideal boundary induced by q_N and π_N^0.[7] The gap between (3) and (1′), or equivalently (1), should be explained in an appropriate way, where (3) seems to be more natural in view of the theorem in 2, Note that (3) and (7) are equivalent with

$$
(13)\quad \begin{aligned} \frac{d}{dt}P^{(f)}(s,x,t,E) &= \int_R P^{(f)}(s,x,t,dy)\left\{\int_R P_{s,t}^{(f)}(dx_1)\pi(x_1\,|\,t,y,E)-\delta_y(E)\right\}, \\ P^{(f)}(s,x,t,E) &\to \delta_x(E), \quad \text{as } t\to s. \end{aligned}
$$

On the other hand, it should be noted that the condition $q(x_1\,|\,t,x) \le q(t,x)$ is so restrictive that a model like the gas of hard balls is excluded, where $q(x_1\,|\,x)$ is proportional to $|x_1-x|$.

A branching model related with (3) will be discussed. Details of proofs and explanations will be published elsewhere.

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6) See also the introduction of [10].

7) See, for instance, Feller [2].

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