75. Absolute Convergence of Fourier Series

By Masako IZUMI and Shin-ichi IZUMI
Department of Mathematics, The Australian National University, Canberra, Australia

(Comm. by Zyoiti SUETUNA, M. J. A., May 12, 1969)

1. Introduction and theorems.

1.1. Let f be an even integrable function, with period 2π and its Fourier series be

(1) $$f(x) \sim \sum_{n=1}^{\infty} a_n \cos nx.$$

R. Mohanty [1] has proved the following

Theorem I. *If* $(I, 1)$ *the function* $\log(2\pi/t)f(t)$ *is of bounded variation on the interval* $(0, \pi)$ *and* $(I, 2)$ *the sequence* $(n^\delta a_n)$ *is of bounded variation for a* $\delta > 0$, *then* $\sum |a_n| < \infty$.

Later one of us [2] proved

Theorem II. *If* $(II, 1)$ f *is of bounded variation and* $\int_0^\pi \log(2\pi/t) \cdot |df(t)| < \infty$ *and* $(II, 2)$ *the sequence* $(n^\delta \Delta(na_n))$ *is of bounded variation for a* $\delta > 0$, *then* $\sum |a_n| < \infty$.

Recently R.M. Mazhar [3] has proved

Theorem III. *If the condition* $(II, 1)$ *is satisfied and* $(III, 2)$ *the sequence*

$$e^{-n^\alpha} \sum_{m=1}^{n} e^{m^\alpha} a_m \qquad (n = 1, 2, \cdots)$$

is of bounded variation for an $\alpha, 0 < \alpha < 1$, *then* $\sum |a_n| < \infty$.

The conditions $(I, 1)$ and $(II, 1)$ are mutually exclusive and $(I, 2)$ and $(II, 2)$ are also. The condition $(III, 2)$ is weaker than $(II, 2)$ ([3], Lemma 2) and then Theorem III is a generalization of Theorem II.

1.2. Our object of this paper is partly to prove Theorem I without using Tauberian theorem and partly to generalize the condition $(I, 1)$as Theorem III, namely:

Theorem 1. *Suppose that the sequence* (m_k) *is positive and increasing and satisfies the following conditions*:

(2) $$m_{k+1}/m_k \leqq A, \ M_k/m_k \leqq Ak^{\delta-\varepsilon} \qquad \text{for an } \varepsilon, 0 < \varepsilon < \delta < 1,$$

where $M_k = m_1 + m_2 + \cdots + m_k$ *and there is an integer* p *such that*

(3) $$|\Delta^{p-1}(M_j \Delta(1/m_j))| \leqq A/j \qquad \text{for all } j > 1.$$

If $(1, 1)$ f *is of bounded variation and* $\int_0^\pi \log(2\pi/t)|df(t)| < \infty$ *and* $(1, 2)$ *the sequence*

$$M_n^{-1}\sum_{k=1}^{n} m_k(k^\delta a_k) \qquad (n=1,2,\cdots)$$

is of bounded variation, then $\sum|a_k|<\infty$.

Theorem 2. *The condition* (1,1) *in Theorem* 1 *can be replaced by the condition* $(I,1)$.

For the proof of Theorems 1 and 2, we use the method in [2].

The sequence $m_k=e^{k^\alpha}$, $1-\delta<\alpha<1$, satisfies the conditions (2) and (3), with $p\geqq 1/(1-\alpha)$.

2. Proof of Theorem 1.

2.1. By (1), $a_n=\dfrac{2}{\pi}\displaystyle\int_0^\pi f(t)\cos nt\,dt$ and then

$$-\frac{\pi}{2}a_n=\frac{1}{n}\int_0^\pi \sin nt\,df(t)=\frac{1}{n}\int_0^{\pi/n^{\delta'}}+\frac{1}{n}\int_{\pi/n^{\delta'}}^{\pi},$$

where $\delta'=\delta'_n$ is taken such that $n^{1-\delta'}$ is an even integer and $\delta'_n\to\sigma$, σ being a small positive number $<\varepsilon/(p-1)$, as $n\to\infty$. We write

$$(4)\qquad \frac{\pi}{2}\sum_{n=1}^{\infty}|a_n|=\sum_{n=1}^{\infty}\frac{1}{n}\left|\int_0^\pi \sin nt\,df(t)\right| \leqq \sum_{n=1}^{\infty}\frac{1}{n}\left|\int_0^{\pi/n^{\delta'}}\right|+\sum_{n=1}^{\infty}\frac{1}{n}\left|\int_{\pi/n^{\delta'}}^{\pi}\right|=P+Q,$$

then we have

$$(5)\qquad P\leqq\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\pi/n^{\delta'}}|df(t)|\leqq\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=[n^{\delta'}]}^{\infty}\int_{\pi/(k+1)}^{\pi/k}|df(t)|$$
$$\leqq A\sum_{k=1}^{\infty}\int_{\pi/(k+1)}^{\pi/k}|df(t)|\sum_{n=1}^{[k^{2/\sigma}]}\frac{1}{n}\leqq A\sum_{k=1}^{\infty}\int_{\pi/(k+1)}^{\pi/k}|df(t)|\log k$$
$$\leqq A\sum_{k=1}^{\infty}\int_{\pi/(k+1)}^{\pi/k}\log\frac{2\pi}{t}\,|df(t)|=A\int_0^\pi\log\frac{2\pi}{t}|df(t)|<\infty$$

by the condition (1,1). It remains to prove that Q is finite.

2.2. By (1) and the assumption (1,1),

$$df(t)\sim\sum_{k=1}^{\infty}k\,a_k\sin kt.$$

Since $\sin nt$ vanishes at the point $t=\pi/n^{\delta'}$, the following Parseval formula [4] holds:

$$(6)\qquad Q_n=\int_{\pi/n^{\delta'}}^{\pi}\sin nt\,df(t)=\sum_{k=1}^{\infty}k\,a_k\int_{\pi/n^{\delta'}}^{\pi}\sin nt\sin kt\,dt.$$

For the sake of simplicity, we put $\delta_n=\pi/n^{\delta'}$, then, by Abel's lemma,

$$(7)\qquad Q_n=\sum_{k=1}^{\infty}(k^\delta a_k)m_k\cdot\frac{k^{1-\delta}}{m_k}\int_{\delta_n}^{\pi}\sin nt\sin kt\,dt$$
$$=\sum_{k=1}^{\infty}A_k\cdot\Delta\left(\frac{k^{1-\delta}}{m_k}\int_{\delta_n}^{\pi}\sin nt\sin kt\,dt\right),$$

where $A_k=\sum_{j=1}^{k}j^\delta a_j m_j$, since, for any fixed n,

$$\left|A_k\cdot\frac{k^{1-\delta}}{m_k}\int_{\delta_n}^{\pi}\sin nt\sin kt\,dt\right|\leqq\frac{A}{k^{\delta}m_k}\left|\sum_{j=1}^{k}j^{\delta}a_jm_j\right|$$
$$=A\left|s_k-\frac{1}{k^{\delta}m_k}\sum_{j=1}^{k-1}s_j((j+1)^{\delta}m_{j+1}-j^{\delta}m_j)\right|\to 0\quad\text{as }k\to\infty,$$

s_k being the kth partial sum of the series $\sum a_j$. Using Abel's lemma again, (7) becomes

$$Q_n=\sum_{k=1}^{\infty}\frac{A_k}{M_k}M_k\cdot\Delta\left(\frac{k^{1-\delta}}{m_k}\int_{\delta_n}^{\pi}\sin nt\sin kt\,dt\right)\tag{8}$$
$$=a_1\sum_{j=1}^{\infty}M_j\cdot\Delta\left(\frac{j^{1-\delta}}{m_j}\int_{\delta_n}^{\pi}\sin nt\sin jt\,dt\right)$$
$$-\sum_{k=1}^{\infty}\Delta\left(\frac{A_k}{M_k}\right)\sum_{j=k+1}^{\infty}M_j\cdot\Delta\left(\frac{j^{1-\delta}}{m_j}\int_{\delta_n}^{\pi}\sin nt\sin jt\,dt\right).$$

In order to see this, we have to prove that

$$\frac{A_k}{M_k}\sum_{j=k+1}^{\infty}M_j\cdot\Delta\left(\frac{j^{1-\delta}}{m_j}\int_{\delta_n}^{\pi}\sin nt\sin jt\,dt\right)\to 0\quad\text{as }k\to\infty\tag{9}$$

for fixed n, which is proved when the series

$$\sum_{j=1}^{\infty}M_j\cdot\Delta\left(\frac{j^{1-\delta}}{m_j}\int_{\delta_n}^{\pi}\sin nt\sin jt\,dt\right)\tag{10}$$

is convergent for each fixed n. For this purpose we shall prove that

$$\sum_{j=N}^{N'}\Delta(j^{1-\delta})\frac{M_j}{m_j}\int_{\delta_n}^{\pi}\sin nt\sin jt\,dt\tag{11}$$
$$+\sum_{j=N}^{N'}(j+1)^{1-\delta}M_j\cdot\Delta(1/m_j)\int_{\delta_n}^{\pi}\sin nt\sin jt\,dt$$
$$+\sum_{j=N}^{N'}\frac{(j+1)^{1-\delta}M_j}{m_{j+1}}\int_{\delta_n}^{\pi}\sin nt(\sin jt-\sin(j+1)t)dt$$
$$=R_{N,N'}+S_{N,N'}+T_{N,N'}\to 0\quad\text{as }N'>N\to\infty.$$

We can suppose $N>n$. Since $n\delta_n$ is even multiple of π,

$$\int_{\delta_n}^{\pi}\sin nt\sin jt\,dt=-\frac{n\sin j\delta_n}{(j-n)(j+n)}\tag{12}$$

and then, by (2),

$$|R_{N,N'}|\leqq A\sum_{j=N}^{N'}\frac{M_j}{j^{2+\delta}m_j}\leqq A\sum_{j=N}^{\infty}\frac{1}{j^{2+\varepsilon}}\to 0\quad\text{as }N\to\infty.\tag{13}$$

Similarly,

$$|S_{N,N'}|\leqq A\sum_{j=N}^{N'}\frac{M_j}{j^{1+\delta}}\Delta\left(\frac{1}{m_j}\right)\leqq A\sum_{j=N}^{\infty}\frac{1}{j^{1+\varepsilon}}\to 0\quad\text{as }N\to\infty.\tag{14}$$

By Abel's lemma,

$$T_{N,N'}=\frac{(N+1)^{1-\delta}M_N}{m_{N+1}}\int_{\delta_n}^{\pi}\sin nt\sin Nt\,dt\tag{15}$$
$$-\sum_{j=N}^{N'-1}\Delta\left(\frac{(j+1)^{1-\delta}M_j}{m_{j+1}}\right)\int_{\delta_n}^{\pi}\sin nt\sin(j+1)t\,dt$$
$$-\frac{(N'+1)^{1-\delta}M_{N'}}{m_{N'+1}}\int_{\delta_n}^{\pi}\sin nt\sin(N'+1)t\,dt$$

and then, by (2),

$$(16)\qquad |T_{N,N'}|\leqq\frac{AM_N}{N^{1+\delta}m_{N+1}}+A\sum_{j=N}^{N'}\left(\frac{M_j}{j^{2+\delta}m_{j+1}}+\frac{1}{j^{1+\delta}}+\frac{M_j}{j^{1+\delta}}\Delta\left(\frac{1}{m_{j+1}}\right)\right)$$
$$+\frac{AM_{N'}}{(N')^{1+\delta}m_{N'+1}}\to 0 \qquad \text{as } N\to\infty.$$

By (11), (13), (14) and (16), the series (10) is convergent, and then the formula (8) holds.

2.3. By (4), (6) and (8)

$$Q=\sum_{n=1}^{\infty}\frac{1}{n}|Q_n|\leqq A\sum_{n=1}^{\infty}\frac{1}{n}\left|\sum_{j=1}^{\infty}M_j\cdot\Delta\left(\frac{j^{1-\delta}}{m_j}\int_{\delta_n}^{\pi}\sin nt\sin jt\,dt\right)\right|$$
$$+A\sum_{k=1}^{\infty}\left|\Delta\left(\frac{A_k}{M_k}\right)\right|\sum_{n=1}^{\infty}\frac{1}{n}\left|\sum_{j=k+1}^{\infty}M_j\cdot\Delta\left(\frac{j^{1-\delta}}{m_j}\int_{\delta_n}^{\pi}\sin nt\sin jt\,dt\right)\right|.$$

By the assumption (1, 2),

$$(17)\qquad Q\leqq A\max_{1\leqq k<\infty}\sum_{n=1}^{\infty}\frac{1}{n}\left|\sum_{j=k}^{\infty}M_j\cdot\Delta\left(\frac{j^{1-\delta}}{m_j}\int_{\delta_n}^{\pi}\sin nt\sin jt\,dt\right)\right|$$
$$\leqq A\max_{1\leqq k<\infty}\sum_{n=1}^{\infty}\frac{1}{n}\left\{\left|\sum_{j=k}^{\infty}\Delta(j^{1-\delta})\frac{M_j}{m_j}\int_{\delta_n}^{\pi}\sin nt\sin jt\,dt\right|\right.$$
$$+\left|\sum_{j=k}^{\infty}(j+1)^{1-\delta}M_j\cdot\Delta\left(\frac{1}{m_j}\right)\int_{\delta_n}^{\pi}\sin nt\sin jt\,dt\right|$$
$$\left.+\left|\sum_{j=k}^{\infty}\frac{(j+1)^{1-\delta}M_j}{m_{j+1}}\int_{\delta_n}^{\pi}\sin nt(\sin jt-\sin(j+1)t)dt\right|\right\}$$
$$\leqq A\max_{1\leqq k<\infty}(R_k+S_k+T_k).$$

Since

$$(18)\qquad \left|\int_{\delta_n}^{\pi}\frac{\sin nt\cos(j\pm 1/2)t}{2\sin t/2}dt\right|\leqq\frac{A}{\delta_n|j-n\pm 1/2|},$$

we have, by (2) and Abel's transformation,

(19)

$$R_k=\sum_{n=1}^{\infty}\frac{1}{n}\left|\sum_{j=k}^{\infty}\Delta(j^{1-\delta})\frac{M_j}{m_j}\int_{\delta_n}^{\pi}\frac{\sin nt}{2\sin t/2}(\cos(j-1/2)-\cos(j+1/2)t)dt\right|$$
$$=\sum_{n=1}^{\infty}\frac{1}{n}\left|\Delta(k^{1-\delta})\frac{M_k}{m_k}\int_{\delta_n}^{\pi}\frac{\sin nt}{2\sin t/2}\cos(k-1/2)t\,dt\right.$$
$$\left.-\sum_{j=k}^{\infty}\Delta\left(\Delta(j^{1-\delta})\frac{M_j}{m_j}\right)\int_{\delta_n}^{\pi}\frac{\sin nt}{2\sin t/2}\cos(j+1/2)t\,dt\right|$$
$$\leqq\sum_{n=1}^{\infty}\frac{A}{n}\left\{\left|\frac{M_k}{k^{\delta}m_k}\frac{1}{\delta_n|k-n-1/2|}\right|\right.$$
$$\left.+\left|\sum_{j=k}^{\infty}\left(\frac{M_j}{j^{1+\delta}m_j}+\frac{1}{j^{\delta}}+\frac{M_j}{j^{\delta}}\Delta\left(\frac{1}{m_j}\right)\right)\frac{1}{\delta_n|j-n+1/2|}\right|\right\}$$
$$=A\frac{\log k}{k^{1+\varepsilon-\sigma}}+A\sum_{j=k}^{\infty}\frac{\log j}{j^{1-\sigma}}\left(\frac{1}{j^{1+\delta}j^{\varepsilon-\delta}}+\frac{1}{j^{\delta}}+\frac{1}{j^{\delta}j^{\varepsilon-\delta}}\right)$$
$$=A\frac{\log k}{k^{1+\varepsilon-\sigma}}+A\sum_{j=k}^{\infty}\frac{\log j}{j^{1+\varepsilon-\sigma}}=A\frac{\log k}{k^{1+\varepsilon-\sigma}}+A\frac{\log k}{k^{\varepsilon-\sigma}}\leqq A.$$

By (2), (3), (17) and $(p-1)$ time use of Abel's lemma,

$$(20)\quad S_k=\sum_{n=1}^{\infty}\frac{1}{n}\left|\sum_{j=k}^{\infty}(j+1)^{1-\delta}M_j\Delta\left(\frac{1}{m_j}\right)\right.$$
$$\left.\cdot\int_{\delta_n}^{\pi}\frac{\sin nt}{2\sin t/2}(\cos(j-1/2)t-\cos(j+1/2)t)dt\right|$$
$$\leqq A\frac{\log k}{k^{\epsilon-\sigma}}+A\frac{\log k}{k^{1-(p-1)\sigma}}\left(\frac{M_k}{k^\delta}+k^{1-\delta}M_k\Delta^2\left(\frac{1}{m_k}\right)+k^{1-\delta}m_{k+1}\Delta\left(\frac{1}{m_{k+1}}\right)\right)$$
$$+A\sum_{j=k}^{\infty}\frac{\log j}{j^{1-(p-1)\sigma}}\left|\Delta^{p-1}\left((j+1)^{1-\delta}M_j\Delta\left(\frac{1}{m_j}\right)\right)\right|$$
$$\leqq A+A\sum_{j=k}^{\infty}\frac{\log j}{j^{1-(p-1)\sigma+\epsilon}}+A\sum_{j=k}^{\infty}\frac{\log j}{j^{\delta-(p-1)\sigma}}\left|\Delta^{p-1}\left(M_j\Delta\left(\frac{1}{m_j}\right)\right)\right|\leqq A.$$

Similarly, by (2), (3), (17) and p time use of Abel's lemma,

$$(21)\quad T_k\leqq A\frac{M_k}{m_k}\frac{\log k}{k^\delta}+A\left(k^{1-\delta}\Delta\left(\frac{M_k}{m_k}\right)+\Delta(k^{1-\delta})\frac{M_{k+1}}{m_{k+1}}\right)\frac{\log k}{k^{1-(p-1)\sigma}}$$
$$+A\sum_{j=k}^{\infty}\frac{\log j}{j^{\delta-(p-1)\sigma}}\Delta^p\left(\frac{M_j}{m_{j+1}}\right)$$
$$\leqq A+A\frac{\log k}{k^{\epsilon-(p-1)\sigma}}+A\sum_{j=k}^{\infty}\frac{\log j}{j^{\delta-(p-1)\sigma}}\Delta^{p-1}\left(M_{j+1}\Delta\left(\frac{1}{m_{j+1}}\right)\right)\leqq A.$$

Collecting (17), (19), (20) and (21), we get $Q\leqq A$, which proves the theorem with (4) and (5).

3. Proof of Theorem 2.

It is sufficient to prove that P in the section 2.1 is finite under the condition $(I, 1)$. Putting $g(t)=\log\frac{2\pi}{t}f(t)$, we get, by integration by parts,

$$P_n=\frac{1}{n}\int_0^{\pi/n^{\delta'}}\sin nt\, d\,f(t)=-\int_0^{\pi/n^{\delta'}}\cos nt\, f(t)\,dt=-\int_0^{\pi/n^{\delta'}}\frac{\cos nt}{\log\frac{2\pi}{t}}g(t)dt$$
$$=-g(\pi/n^{\delta'})\int_0^{\pi/n^{\delta'}}\frac{\cos nu}{\log\frac{2\pi}{u}}du+\int_0^{\pi/n^{\delta'}}\left(\int_0^t\frac{\cos nu}{\log\frac{2\pi}{u}}du\right)dg(t)$$
$$=-g(\pi/n^{\delta'})\frac{1}{n}\int_0^{\pi/n^{\delta'}}\frac{\sin nu}{u\left(\log\frac{2\pi}{u}\right)^2}du+\int_0^{\pi/n^{\delta'}}\left(\int_0^t\frac{\cos nu}{\log\frac{2\pi}{u}}du\right)dg(t).$$

Thus we have

$$|P_n|\leqq\frac{A}{n(\log n)^2}+\frac{A}{n}\int_0^{\pi/n^{\delta'}}\frac{|dg(t)|}{\log\frac{2\pi}{t}},$$

since

$$\int_0^{\pi/n^{\delta'}}\frac{\sin nu}{u\left(\log\frac{2\pi}{u}\right)^2}du=\int_0^{\pi/n}+\int_{\pi/n}^{\pi/n^{\delta'}}=O\left(\frac{1}{(\log n)^2}\right)$$

by the mean value theorem, and then

$$P=\sum_{n=1}^{\infty}|P_n|\leqq A\sum_{n=2}^{\infty}\frac{1}{n(\log n)^2}+A\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=[n^{\delta'}]}^{\infty}\int_{\pi/(k+1)}^{\pi/k}\frac{|dg(t)|}{\log\dfrac{2\pi}{t}}$$

$$\leqq A+\int_0^{\pi}|dg(t)|<A.$$

Thus we get Theorem 2.

References

[1] R. Mohanty: A criterion for the absolute convergence of a Fourier series. Proc. London Math. Soc., **51** (2), 186–196 (1949).
[2] S. Izumi: Absolute convergence of some trigonometric series. II. Jour. of Math. Analysis and Applications, **1**, 184–193 (1960).
[3] R. M. Mazhar: Absolute convergence of Fourier series. Proc. Japan Acad., **44**, 756–761 (1968).
[4] A. Zygmund: Trigonometric Series. I. Cambridge Press (1959).