132. *On Infinitesimal Affine Automorphisms of Siegel Domains*

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A non-empty open cone V in a finite dimensional vector space X over $\boldsymbol{R}$ is called a *convex cone*, if it is convex and contains no straight lines. For example, the cone $\mathcal{P}(m, \boldsymbol{R})$ ($\mathcal{P}(m, \boldsymbol{C})$) of all positive-definite real symmetric (complex hermitian) matrices of degree m is a convex cone. For a convex cone V in X, an $\boldsymbol{R}$-bilinear map F on a finite dimensional vector space Y over $\boldsymbol{C}$ into the complexification X^C of X is called a *V-hermitian function* if it is $\boldsymbol{C}$-linear with respect to the first variable and $F(u, v)=\overline{F(v, u)}$, where $z\mapsto\bar{z}$ is the conjugation of X^C with respect to the real form X, and if it is V-positive-definite in the sense that $F(u, u)\in\bar{V}$ (the closure of V in X) and $F(u, u)=0$ implies $u=0$ for $u\in Y$. For a V-hermitian function F, the domain $D(V, F)=\{(z, u)\in X^C\times Y\,;\,\mathcal{I}m z-F(u, u)\in V\}$ of $X^C\times Y$ is called a *Siegel domain* associated to V and F. A Siegel domain $D(V, F)$ in $X^C\times Y$ is called *irreducible* if Y is not the direct sum of two non-trivial subspaces which are mutually orthogonal with respect to F. For Siegel domains $D(V, F)\subset X^C\times Y$ and $D(V', F')\subset X'^C\times Y'$, an affine isomorphism φ of $X^C\times Y$ onto $X'^C\times Y'$ is called an *affine isomorphism* of $D(V, F)$ onto $D(V', F')$ if $\varphi(D(V, F))=D(V', F')$. An affine isomorphism of a Siegel domain $D(V, F)$ onto itself is called an *affine automorphism* of $D(V, F)$. If the group of affine automorphisms of a Siegel domain is transitive on it the domain is said to be *homogeneous*.

In this note we shall state a theorem which reduces the classification of homogeneous Siegel domains with respect to affine isomorphism to the one of certain distributive algebras over $\boldsymbol{R}$ and we shall describe the structure of the Lie algebra of the group of affine automorphisms of a homogeneous Siegel domain in terms of the above algebra.

A finite dimensional distributive algebra $\mathfrak{C}$ over $\boldsymbol{R}$ is called a *matrix algebra with involution* * of rank $m+1$ if : 1) it is bigraded: $\mathfrak{C}=\sum_{1\le i,k\le m+1}\mathfrak{C}_{ik}$, 2) $\mathfrak{C}_{ik}\mathfrak{C}_{kl}\subset\mathfrak{C}_{il}$, $\mathfrak{C}_{ik}\mathfrak{C}_{pq}=\{0\}$ if $k\neq p$, 3) $a\mapsto a^*$ is an involutive anti-automorphism of the algebra $\mathfrak{C}$, $\mathfrak{C}_{ik}^*=\mathfrak{C}_{ki}$, 4) if we put $n_{ik}=\dim\mathfrak{C}_{ik}$, we have $n_{ii}\neq 0$ for $1\le i\le m+1$. Henceforce, a_{ik}, b_{ik}, $\cdots$ will always denote arbitrary elements of the subspace $\mathfrak{C}_{ik}$. A matrix algebra $\mathfrak{C}$ with involution of rank $m+1$ is called an *S-algebra* of rank m if $\mathfrak{C}_{i,m+1}$, $\mathfrak{C}_{m+1,i}$ $(1\leq i\leq m)$ have complex structures j $(j^2=-1)$ and 1) there exists an algebra isomorphism ρ of $\mathfrak{C}_{ii}$ onto $\boldsymbol{R}$ for each i such that $1\leq i\leq m+1$, 2) $a_{ii}b_{ik}=\rho(a_{ii})b_{ik}$, 3) there exists a positive number n_i for each $i: 1\leq i\leq m+1$ such that $n_i\rho(a_{ik}b_{ki})=n_k\rho(b_{ki}a_{ik})$, 4) $\rho(a_{ik}a^*_{ik})>0$ if $a_{ik}\neq 0$, 5) $a_{ik}(b_{kl}c_{li})=(a_{ik}b_{kl})c_{li}$, 6) $a_{ik}(b_{kl}c_{lp})=(a_{ik}b_{kl})c_{lp}$ if $i<k<l$ and $k<p$, 7) $a_{ik}(b_{kl}b^*_{kl})=(a_{ik}b_{kl})b^*_{kl}$ if $i<k<l$, 8) $a_{ik}(jb_{k,m+1})=j(a_{ik}b_{k,m+1})$ if $i<k<m+1$, 9) $jb^*_{i,m+1}=(jb_{i,m+1})^*$ for $1\leq i\leq m$, 10) $\rho(jb_{i,m+1}(jb_{i,m+1})^*)=\rho(b_{i,m+1}b^*_{i,m+1})$ for $1\leq i\leq m$.

Let $\mathfrak{C}$ be an S-algebra of rank m and let $i\mapsto i'$ be a permutation of indices $1, 2, \cdots, m$. The change in the grading of $\mathfrak{C}$ which renames the subspaces $\mathfrak{C}_{ik}$, $\mathfrak{C}_{i,m+1}$, $\mathfrak{C}_{m+1,i}$ into $\mathfrak{C}_{i'k'}$, $\mathfrak{C}_{i',m+1}$, $\mathfrak{C}_{m+1,i'}$, respectively, is called *inessential* if $i<k$ and $i'>k'$ imply $n_{ik}=0$. Two *isomorphic S-algebras* are defined as such if they have the same rank and after an inessential change in the grading they become isomorphic as bigraded algebras with involution and partial complex structure.

For an S-algebra $\mathfrak{C}$ of rank m, we put $\mathfrak{A}=\sum_{1\leq i,k\leq m}\mathfrak{C}_{ik}$, $\mathfrak{B}=\sum_{1\leq i\leq m}(\mathfrak{C}_{i,m+1}+\mathfrak{C}_{m+1,i})$, $\mathfrak{X}=\{a\in\mathfrak{A}\,;\,a^*=a\}$ and $\mathfrak{Y}=\{b\in\mathfrak{B}\,;\,b^*=b\}$. Then $\mathfrak{Y}$ may be considered as a vector space over $\boldsymbol{C}$ by the complex structure j. In the following, a_{ik} will denote the $\mathfrak{C}_{ik}$-component of $a\in\mathfrak{C}$. Let $V(\mathfrak{A})=\{tt^*\,;\,t\in\sum_{1\leq i\leq k\leq m}\mathfrak{C}_{ik},\ \rho(t_{ii})>0 \text{ for } 1\leq i\leq m\}$. We define maps $\Phi:\mathfrak{Y}\times\mathfrak{Y}\to\mathfrak{X}$ and $F:\mathfrak{Y}\times\mathfrak{Y}\to\mathfrak{X}^C$ by $\Phi(u,v)=\dfrac{1}{2}\sum_{1\leq i,k\leq m}(u_{i,m+1}v_{m+1,k}+v_{i,m+1}u_{m+1,h})$ and $F(u,v)=\dfrac{1}{2}(\Phi(u,v)+i\Phi(u,jv))$. Then it is proved that $V(\mathfrak{A})$ is a convex cone in $\mathfrak{X}$ (Vinberg [4]) and F is a $V(\mathfrak{A})$-hermitian function. $D(\mathfrak{C})$ denotes the Siegel domain in $\mathfrak{X}^C\times\mathfrak{Y}$ associated to $V(\mathfrak{A})$ and F. Then by means of the theory of Vinberg [4] on T-algebras which are, by definition, matrix algebras with involution including properties 1)–7), together with the theory of Pjateckii-Šapiro [2] on normal j-algebras, we have:

Theorem A. $\mathfrak{C}\to D(\mathfrak{C})$ *gives a bijective correspondence of the set of isomorphism classes of S-algebras with the set of affine isomorphism classes of homogeneous Siegel domains.*

Example 1. Let m and n be integers such that $0\leq n\leq m$. Let $\mathfrak{C}_{ik}=\boldsymbol{R}$ for $1\leq i, k\leq m$, $\mathfrak{C}_{m+1,m+1}=\boldsymbol{R}$, $\mathfrak{C}_{i,m+1}=\{0\}$, $\mathfrak{C}_{m+1,i}=(0)$ for $n+1\leq i\leq m$, $\mathfrak{C}_{i,m+1}=C$, $\mathfrak{C}_{m+1,i}=C$, j the natural complex structure on $\mathfrak{C}_{i,m+1}=\boldsymbol{C}$ for $1\leq i\leq n$. Then an element x of $\mathfrak{C}=\sum_{1\leq i,k\leq m+1}\mathfrak{C}_{ik}$ is considered as a square matrix of degree $m+1$ with coefficients in $\boldsymbol{C}$. The involution is defined by $x^*={}^t\bar{x}$. Let $\rho=$identity. We define the multiplication of the "upper nilpotent part" of $\mathfrak{C}$ by the ordinary matrix multiplication. The other multiplication is automatically determined from the former one by the requirement that $(ab, c)=(a, cb^*)=(b, a^*c)$, where $(\ ,\ )$ is the natural Euclidean metric on $\mathfrak{C}$. Then $\mathfrak{C}$ is an S-algebra and $D(\mathfrak{C})$ is a Siegel domain with $V=\mathcal{P}(m, \boldsymbol{R})$.

Example 2. (I) Let m and n be as the above. The underlying linear space $\mathfrak{C}$, ρ and j are defined similarly to the above except that $\mathfrak{C}_{ik}=\boldsymbol{C}$ for $1\leq i, k\leq m, i\neq k$. The involution is defined in the same way as above. We define two ways of multiplication of the "upper nilpotent part" of $\mathfrak{C}$, so that we have two ways of multiplication of $\mathfrak{C}$ using the same method as above. $(\mathrm{I})_0$: The multiplication is the ordinary matrix multiplication. $(\mathrm{I})_1$: The multiplication is the ordinary matrix multiplication except that the product of $\lambda\in\mathfrak{C}_{ik}=\boldsymbol{C}$ $(1\leq i<k\leq n)$ and $\mu\in\mathfrak{C}_{k,m+1}=\boldsymbol{C}$ is $\bar{\lambda}\mu\in\mathfrak{C}_{i,m+1}=\boldsymbol{C}$. Then $\mathfrak{C}$ is an S-algebra in both cases.

(II) Let m and n be integers such that $1\leq n\leq m-1$ and θ a real number such that $0<\theta<\frac{\pi}{2}$. The underlying linear spaces of $\mathfrak{A}=\sum_{1\leq i,k\leq m}\mathfrak{C}_{ik}$, $\mathfrak{C}_{m+1,m+1}$, $\mathfrak{C}_{i,m+1}$, $\mathfrak{C}_{m+1,i}$ $(n+1\leq i\leq m)$ and ρ are the same as (I). Let $\mathfrak{C}_{i,m+1}=\boldsymbol{C}\times\boldsymbol{C}$, $\mathfrak{C}_{m+1,i}=\boldsymbol{C}\times\boldsymbol{C}$ for $1\leq i\leq n-1$, $\mathfrak{C}_{n,m+1}=\boldsymbol{C}$, $\mathfrak{C}_{m+1,n}=\boldsymbol{C}$ and j the natural complex structure on $\mathfrak{C}_{i,m+1}=\boldsymbol{C}\times\boldsymbol{C}$ or on $\mathfrak{C}_{n,m+1}=\boldsymbol{C}$. The involution is defined in the same way as (I). The multiplication of the "upper nilpotent part" of $\mathfrak{C}$ is the ordinary matrix multiplication except the following: denoting canonical basis $1\in\mathfrak{C}_{ik}=\boldsymbol{C}$ $(1\leq i<k\leq n)$, $(1, 0)$, $(0, 1)\in\mathfrak{C}_{i,m+1}=\boldsymbol{C}\times\boldsymbol{C}$ $(1\leq i\leq n-1)$, $1\in\mathfrak{C}_{n,m+1}=\boldsymbol{C}$ by e_{ik}, $e^{(1)}_{i,m+1}$, $e^{(2)}_{i,m+1}$, $e_{n,m+1}$, respectively, we define for $\lambda\in\boldsymbol{C}$ that $(\lambda e_{ik})e^{(1)}_{k,m+1}=\lambda e^{(1)}_{i,m+1}$, $(\lambda e_{ik})e^{(2)}_{k,m+1}=\bar{\lambda}e^{(2)}_{i,m+1}$ for $1\leq i<k<n-1$ and $(\lambda e_{in})(e_{n,m+1})=\cos\theta\lambda e^{(1)}_{i,m+1}+\sin\theta\bar{\lambda}e^{(2)}_{i,m+1}$ for $1\leq i\leq n-1$. Then $\mathfrak{C}$ is an S-algebra.

In both cases (I) and (II), $D(\mathfrak{C})$ is a Siegel domain with $V=\mathcal{P}(m, \boldsymbol{C})$.

Remark. It can be proved that any irreducible homogeneous Siegel domain with $V=\mathcal{P}(m, \boldsymbol{R})$ or $\mathcal{P}(m, \boldsymbol{C})$ is affinely isomorphic with one of Siegel domains obtained in Example 1 or Example 2.

Now we want to describe the structure of the Lie algebra $\mathfrak{g}$ of the group of affine automorphisms of $D(\mathfrak{C})$ for an S-algebra $\mathfrak{C}$ of rank m. We define three spaces of infinitesimal affine automorphisms of $\mathfrak{X}^C\times\mathfrak{Y}$ as follows:

$$\mathfrak{x}=\{D_\xi: (z, u)\mapsto(\xi, 0)\ (z\in\mathfrak{X}^C, u\in\mathfrak{Y});\ \xi\in\mathfrak{X}\},$$
$$\mathfrak{y}=\{D_\eta: (z, u)\mapsto(2iF(u, \eta), \eta)\ (z\in\mathfrak{X}^C, u\in\mathfrak{Y});\ \eta\in\mathfrak{Y}\},$$
$$\mathfrak{h}=\{D=(D_{\mathfrak{X}}, D_{\mathfrak{Y}})\in\mathfrak{gl}V(\mathfrak{A})\times\mathfrak{gl}(\mathfrak{Y});D_{\mathfrak{Y}}j=jD_{\mathfrak{Y}},$$
$$D_{\mathfrak{X}}\Phi(u, v)=\Phi(D_{\mathfrak{Y}}u, v)+\Phi(u, D_{\mathfrak{Y}}v)\ (u, v\in\mathfrak{Y})\},$$

where $\mathfrak{gl}V$ denotes the Lie algebra of the group of linear automorphisms of X leaving the cone V in X invariant. Then (Pjateckii-Šapiro [1]) we have $\mathfrak{g}=\mathfrak{h}+\mathfrak{x}+\mathfrak{y}$ and $[\mathfrak{x}, \mathfrak{x}]=\{0\}$, $[\mathfrak{x}, \mathfrak{y}]=\{0\}$, $[D_\eta, D_{\eta'}]=D_{2\Phi(\eta, j\eta')}$ for

$\eta, \eta' \in \mathfrak{Y}$, $[D, D_\xi]=D_{D_{\mathfrak{X}}\xi}\ (\xi \in \mathfrak{X})$, $[D, D_\eta]=D_{D_{\mathfrak{Y}}\eta}\ (\eta \in \mathfrak{Y})$ for $D=(D_{\mathfrak{X}}, D_{\mathfrak{Y}}) \in \mathfrak{h}$, so that we may restrict ourselves to determine the structure of the Lie algebra $\mathfrak{h}$.

We can define an equivalence relation "$\sim$" in indices $\{1, 2, \cdots, m\}$ modifying the one of Vinberg [5] in such a way that after an inessential change in the grading we have: (A) $i\sim k$ implies $n_{ik}\neq 0$, $n_{il}=n_{kl}$ for l such that $l\neq i$, $l\neq k$, $1\leq l\leq m$ and $n_{i,m+1}=n_{k,m+1}$, (B) $i<k<l$ and $i\sim l$ imply $i\sim k\sim l$. In that which follows we shall assume that these conditions hold. Let $\bigcup_{1\leq\alpha\leq\mu} M_\alpha$ be the decomposition of $\{1, 2, \cdots, m\}$ into $\sim$-equivalence classes. If we put $\mathfrak{A}_\alpha=\sum_{i,k\in M_\alpha}\mathfrak{C}_{ik}$, $\mathfrak{X}_\alpha=\mathfrak{X}\cap\mathfrak{A}_\alpha$, then the subset $V(\mathfrak{A}_\alpha)$ of $\mathfrak{X}_\alpha$ defined in the same way as $V(\mathfrak{A})$ is a *self-dual* convex cone in $\mathfrak{X}_\alpha$, that is, there exists a Euclidean metric $(\ ,\ )$ on $\mathfrak{X}_\alpha$ such that $V(\mathfrak{A}_\alpha)=\{x\in\mathfrak{X}_\alpha;\ (x, \overline{V(\mathfrak{A}_\alpha)}-\{0\})>0\}$ (Vinberg [5]). We put $\mathfrak{C}^c=\sum_{1\leq\alpha\leq\mu}\mathfrak{A}_\alpha$, $\mathfrak{X}^c=\mathfrak{X}\cap\mathfrak{C}^c$ and $V^c=\sum_{1\leq\alpha\leq\mu}V(\mathfrak{A}_\alpha)$. A bigraded endomorphism D of $\mathfrak{C}$ is called a *j-derivation* of $\mathfrak{C}$ if $D(ab)=(Da)b+a(Db)$, $Da^*=(Da)^*$ $(a, b\in\mathfrak{C})$, $D\mathfrak{C}_{m+1,m+1}=\{0\}$ and $Dj=jD$ on $\mathfrak{B}$. Note that any j-derivation of $\mathfrak{C}$ leaves $\mathfrak{X}\times\mathfrak{Y}$ invariant. $\mathscr{D}_j(\mathfrak{C})_0$ denotes the Lie algebra of all j-derivations D with $D\mathfrak{C}^c=\{0\}$ and $\mathfrak{g}_0$ the Lie algebra of all $D\in\mathfrak{h}$ with $D\mathfrak{X}^c=\{0\}$. Put $\mathfrak{T}^u=\sum_{\alpha<\beta}\sum_{i\in M_\alpha, k\in M_\beta}\mathfrak{C}_{ik}$ and $\mathfrak{t}^u=\{D_t: (x, y)\mapsto(tx+xt^*, ty+yt^*)\ (x\in\mathfrak{X}, y\in\mathfrak{Y});\ t\in\mathfrak{T}^u\}$. Then we have:

Theorem B. *The Lie algebra* $\mathfrak{h}$ *has the decomposition*:

$$\mathfrak{h}=\mathfrak{g}_0+\mathfrak{g}^c+\mathfrak{t}^u$$

with the following properties:

1) $\mathfrak{g}^c$ *is a reductive subalgebra without compact factors and leaves* $\mathfrak{X}^c$ *invariant. The restriction of* $\mathfrak{g}^c$ *to* $\mathfrak{X}^c$ *induces an isomorphism of* $\mathfrak{g}^c$ *onto* $\mathfrak{gl}V^c$.

2) $\mathfrak{g}_0$ *is a compact subalgebra. The restriction of* $\mathscr{D}_j(\mathfrak{C})_0$ *to* $\mathfrak{X}\times\mathfrak{Y}$ *induces an isomorphism of* $\mathscr{D}_j(\mathfrak{C})_0$ *onto* $\mathfrak{g}_0$.

3) $\mathfrak{t}^u$ *is an* $\boldsymbol{R}$*-triangular nilpotent ideal.*

4) $[\mathfrak{g}_0, \mathfrak{g}^c]=\{0\}$, $[D, D_t]=D_{Dt}$ *for* $D\in\mathfrak{g}_0=\mathscr{D}_j(\mathfrak{C})_0$, $t\in\mathfrak{T}^u$, $[D_t, D_{t'}]=D_{tt'-t't}$ *for* $t, t'\in\mathfrak{T}^u$ *and the action of* $\mathfrak{g}^c$ *on* $\mathfrak{t}^u$ *or on* $\mathfrak{X}\times\mathfrak{Y}$ *can be described in terms of the completion of* $\mathfrak{C}^c$ *due to Vinberg* [5].

Example. Let $\mathfrak{C}$ be the S-algebra of Example 1. Then $\mathfrak{A}$ is the Lie algebra $\mathfrak{gl}(m, \boldsymbol{R})$ of matrices with coefficients in $\boldsymbol{R}$ of degree m with the rule $[a, b]=ab-ba$.

$$\mathfrak{T}^u=\left\{\begin{pmatrix}\overbrace{0}^{n} & \overbrace{b}^{m-n}\\ 0 & 0\end{pmatrix};\ b\quad n\times(m-n)\text{ matrix with coefficients in }\boldsymbol{R}\right\}.$$

Let $a\mapsto D_a$ denote the homomorphism of $\mathfrak{gl}(m, \boldsymbol{R})$ into $\mathfrak{gl}(\mathfrak{X}\times\mathfrak{Y})$ defined by $D_a: (x, y)\mapsto(ax+xa^*, ay+ya^*)$. Then $\mathfrak{g}^c$ is isomorphic with the naturally imbedded subalgebra $\mathfrak{gl}(n, \boldsymbol{R})\times\mathfrak{gl}(m-n, \boldsymbol{R})$ of $\mathfrak{gl}(m, \boldsymbol{R})$: $\mathfrak{g}^c$

$=\{D_a\,;\,a\in\mathfrak{gl}(n,\boldsymbol{R})\times\mathfrak{gl}(m-n,\boldsymbol{R})\}$, $\mathfrak{t}^u=\{D_t\,;\,t\in\mathfrak{T}^u\}$ and $[D_a, D_t]=D_{[a,t]}$ for $a\in\mathfrak{gl}(n,\boldsymbol{R})\times\mathfrak{gl}(m-n,\boldsymbol{R})$, $t\in\mathfrak{T}^u$. If $n=0$, then $\mathfrak{g}_0=\{0\}$. If $n>0$, then $\mathfrak{g}_0\cong\mathfrak{o}(2):\mathfrak{g}_0=\{(x,y)\mapsto(0,i\lambda y)\ (x\in\mathfrak{X}, y\in\mathfrak{Y}=\boldsymbol{C}^n)\,;\,\lambda\in\boldsymbol{R}\}$ and therefore $[\mathfrak{g}_0, \mathfrak{t}^u]=\{0\}$ (cf. Tanaka [3]).

Application. Recently Tanaka [3] has described the structure of the Lie algebra of infinitesimal holomorphic automorphisms of a homogeneous Siegel domain in terms of the prolongation in his sense of the pair $(\mathfrak{x}+\mathfrak{y}, \mathfrak{h})$ of the graded Lie algebra $\mathfrak{x}+\mathfrak{y}$ and the Lie algebra $\mathfrak{h}$ of derivations of $\mathfrak{x}+\mathfrak{y}$. On the other hand it is known [6] that any homogeneous bounded domain can be realized uniquely as a homogeneous Siegel domain. Thus by using Theorem B we can find all infinitesimal automorphisms of a homogeneous bounded domain.

References

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