125. *A Note on Two Inequalities Correlated to Unitary ρ-Dilatations*

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1. In [7] Sz-Nagy and C. Foias introduced the notion of the class $\mathcal{C}_\rho$ as follows. For each fixed $\rho>0$, $\mathcal{C}_\rho$ is the class of operators T on a given complex Hilbert space H having the following property:

There exist a Hilbert space K containing H as a subspace and a unitary operator U on K satisfying the following representation

(*) $$T^n=\rho PU^n \ (n=1, 2, \cdots)$$

where P is the orthogonal projection of K on H.

It is well known that $\mathcal{C}_1=\{T: \|T\|\leqq 1\}$ ([6]) and $\mathcal{C}_2=\{T: \|T\|_N\leqq 1\}$ ([1]) where $\|T\|_N$ means the numerical radius of T,

$$\|T\|_N=\sup|(Th, h)| \quad \text{for every unit vector } h \text{ in } H.$$

The following theorem is known and we cite for the sake of convenience ([5] [7]).

Theorem A. (i) *For each fixed* $\rho>0$ *and* T *on* $\mathcal{L}(H)$, $T\in\mathcal{C}_\rho$ *if and only if*

$$\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}T_\rho(n)\geqq 0 \quad \textit{for every } \theta \textit{ and } r \textit{ such that } 0\leqq r<1.$$

$$\textit{where} \quad T_\rho(n)=\begin{cases} \dfrac{1}{\rho}T^n & (n\geqq 1) \\ I & (n=0) \\ \dfrac{1}{\rho}T^{*-n} & (n\leqq -1) \end{cases}$$

(ii) $\mathcal{C}_\rho$ *is non-decreasing with respect to the index* ρ *in the sense that* $\mathcal{C}_{\rho_1}\subset\mathcal{C}_{\rho_2}$ *if* $0<\rho_1<\rho_2$.

In [5] J. Holbrook has defined the function w_ρ as follows

$$w_\rho(T)=\inf\left\{u\,;\,u>0 \quad \frac{1}{u}T\in\mathcal{C}_\rho\right\}.$$

Concerning to this function $w_\rho(T)$ he has proved the following theorems

Theorem B. $w_\rho(T)$ *has the following properties*:

(1) $$w_\rho(T)<\infty$$

(2) $$w_\rho(T)>0 \textit{ unless } T=0;\ \textit{in fact } w_\rho(T)\geqq\frac{1}{\rho}\|T\|$$

(3) $$w_\rho(zT)=|z|\,w_\rho(T)$$

(4) $$w_\rho(T)\leqq 1 \textit{ if and only if } T\in\mathcal{C}_\rho$$

(5) $$w_\rho(T) \textit{ is a norm whenever } 0<\rho\leqq 2$$

(6) *$w_\rho(T)$ is continuous and non-increasing as a function of ρ.*

Theorem C. *For each $\rho>0$ and $T\in\mathcal{L}(H)$, then*

$$w_\rho(T^k)\leqq(w_\rho(T))^k \qquad (k=1, 2, \cdots\cdots).$$

Theorem D. *For any $T\in\mathcal{L}(H)$ and $\rho>0$,*

$$K_\rho r(T)\leqq w_\rho(T)\leqq K_\rho\|T\|$$

where $r(T)$ means the spectral radius of T and $K_\rho=\begin{cases}1 & (\rho\geqq 1)\\ \dfrac{2}{\rho}-1 & (0<\rho<1).\end{cases}$

The first purpose of this paper is to show that we cannot form the class of "*ρ-loid*" *operators* which is included in the class of normaloids by using this function $w_\rho(T)$ in consequence of Theorem 1. The second purpose is to give Theorem 4 as a sharpening of Theorem D.

We should like to express here our cordial thanks to Professor Z. Takeda for his kind advice in the preparation of this paper.

2. Theorem 1. *For each $0<\rho<1$ and $T\in\mathcal{L}(H)$, then there does not exist an operator T unless $T=0$ satisfying the following equality*

$$w_\rho(T^k)=(w_\rho(T))^k \quad (k=1, 2, \cdots).$$

Proof. We assume that

$$(**) \qquad w_\rho(T^k)=(w_\rho(T))^k \quad (k=1, 2, \cdots\cdots)$$

in Theorem C. By Theorem D we have

$$K_\rho r(T^k)\leqq w_\rho(T^k)\leqq K_\rho\|T^k\|$$

where

$$K_\rho=\frac{2}{\rho}-1>1 \quad \text{for } 0<\rho<1.$$

By (**) and the spectral mapping theorem, we have

$$K_\rho r(T)^k\leqq(w_\rho(T))^k\leqq K_\rho\|T^k\|$$

$$K_\rho^{\frac{1}{k}} r(T)\leqq w_\rho(T)\leqq K_\rho^{\frac{1}{k}}\|T^k\|^{\frac{1}{k}}$$

so that we get the following inequality as $k\to\infty$

$$r(T)\leqq w_\rho(T)\leqq r(T)$$

consequently we have

$$K_\rho r(T)\leqq w_\rho(T)=r(T)<K_\rho r(T)$$

where $K_\rho=\dfrac{2}{\rho}-1>1$ for $0<\rho<1$. This contradiction proves Theorem 1.

3. By Theorems C and D we have the following properties for $\rho\geqq 1$

(7) intermediate property; $r(T)\leqq w_\rho(T)\leqq\|T\|$

(8) power inequality; $w_\rho(T^k)\leqq(w_\rho(T))^k \quad (k=1, 2, \cdots\cdots)$.

In the preceding paper [4], we have defined for every unit vector x and y.

Definition 1. $\|T\|_p=\sup|(T^px, y)|^{\frac{1}{p}}=\|T^p\|^{\frac{1}{p}}$ for any integer p.

Definition 2. $\|T\|_{Np}=\sup|(T^px, x)|^{\frac{1}{p}}=\|T^p\|_N^{\frac{1}{p}}$ for any integer p.

These satisfy the intermediate property (7) and the power inequality (8), hence we have given the following theorem by these useful properties ([4]).

Theorem E.

(i) $\|T\|_p = r(T)$ *if and only if* $\|T^n\|_p = \|T\|_p^n$ $(n=1, 2, \cdots\cdots)$

(ii) $\|T\|_{Np} = r(T)$ *if and only if* $\|T^n\|_{Np} = \|T\|_{Np}^n$ $(n=1, 2, \cdots\cdots)$.

We put $\mathcal{N} = \{N_p ; p=1, 2, \cdots\}$ where $N_p = \{T ; \|T\|_p = r(T)\}$

$\mathcal{S} = \{S_p ; p=1, 2, \cdots\}$ where $S_p = \{T ; \|T\|_{Np} = r(T)\}$

Definition 3. An operator T is said to be normaloid if $\|T^n\| = \|T\|^n$ $(n=1, 2, \cdots)$, spectraloid if $\|T^n\|_N = \|T\|_N^n$ $(n=1, 2, \cdots)$.

Theorem F. *The family of classes $\mathcal{N} = \{N_p ; p=1, 2, \cdots\}$ forms an atomic lattice by the set inclusion relation. The greatest element is the whole set of operators N_∞, the least element is the class of normaloids and the atomic elements are N_p's with prime indices.*

Theorem G. *The family of classes $\mathcal{S} = \{S_p ; p=1, 2, \cdots\}$ forms an atomic lattice by the set inclusion relation. The greatest element is the whole set of operators S_∞, the least element is the class of spectraloids, and the atomic elements are S_p's with prime indices.*

In connection with the above theorems we shall define $\mathcal{L}$ as follows

Definition 4. $L_\rho = \{T ; w_\rho(T) = r(T)$ for $\rho \geqq 1\}$

$\mathcal{L} = \{L_\rho ; \rho \geq 1\}$.

By using the analogous argument in [4], we have the following theorem

Theorem 2. *For each $\rho \geqq 1$, then*

$$w_\rho(T) = r(T) \quad \textit{if and only if } w_\rho(T^n) = (w_\rho(T))^n \quad (n=1, 2, \cdots\cdots).$$

Definition 5. An operator T is named to be "ρ-loid" operator if

$$w_\rho(T^k) = (w_\rho(T))^k \quad (k=1, 2, \cdots\cdots)$$

1-loid and 2-loid are normaloid and spectraloid respectively.

As $w_\rho(T)$ is continuous and non-increasing as a function of ρ, we have

Theorem 3. *The family of classes $\mathcal{L} = \{L_\rho ; \rho \geqq 1\}$ forms a monotone non-decreasing family as a function of ρ by the set inclusion relation. The greatest element is the whole set of operators L_∞, the least element is the class of normaloids L_1 and L_2 means the class of spectraloids.*

At the end of this section we state an application of Theorem 1. As $w_\rho(T)$ is a non-increasing function of ρ and (2) of Theorem B, then

$$w_1(T) = \|T\| < w_\rho(T) \quad \text{for } 0 < \rho < 1$$

so that we may naturally come to mind the following question;

Can we form the class of "ρ-loid" operators which is included in the class of normaloids by using this $w_\rho(T)$?

But the answer is *not affirmative*, because by Theorem 1 there does not exist an operator T unless $T=0$ satisfying the following equality

$$w_\rho(T^k)=(w_\rho(T))^k \quad (k=1, 2, \cdots\cdots)$$

for each $0<\rho<1$ and $T \in \mathcal{L}(H)$.

4. In this section we shall give a sharpening of Theorem D.

Theorem 4. *For any* $T \in \mathcal{L}(H)$ *and* $\rho>0$, *then*

$$K_\rho\|T\| \geqq w_\rho(T) \geqq \begin{cases} K_\rho w_2(T) & (0<\rho<1) \\ w_2(T) & (1\leqq\rho\leqq 2) \\ r(T) & (2<\rho). \end{cases}$$

We cite the following lemma without proof.

Lemma ([2] [3]). $S \in C_\rho$ *implies* $1 \geqq \begin{cases} K_\rho w_2(S) & (0<\rho<1) \\ w_2(S) & (1\leqq\rho\leqq 2). \end{cases}$

Proof of Theorem 2. We have only to prove Theorem 4 in the case $0\leqq\rho\leqq 2$. When $w_\rho(T)=0$, then $T=0$ so that this case is trivial. We may assume $w_\rho(T)\neq 0$.

For any operator T on $\mathcal{L}(H)$ we put $S=\dfrac{T}{w_\rho(T)}$, then $w_\rho(S)=1$ so $S \in C_\rho$ by Theorem B. By Lemma we have

$$1=w_\rho\left(\frac{T}{w_\rho(T)}\right) \geqq \begin{cases} K_\rho w_2\left(\dfrac{T}{w_\rho(T)}\right) & (0<\rho<1) \\ w_2\left(\dfrac{T}{w_\rho(T)}\right) & (1\leqq\rho\leqq 2). \end{cases}$$

By the homogeneity of $w_\rho(T)$ the proof is complete.

In general we have $w_2(T) \geqq r(T)$, thus Theorem 4 is a sharpening of Theorem D.

References

[1] C. Berger: On the numerical range of an operator (to appear).
[2] T. Furuta: Relations between unitary ρ-dilatations and two norms. Proc. Japan Acad., **44**, 16–20 (1968).
[3] ——: Relations between unitary ρ-dilatations and two norms. II. Proc. Japan Acad., **44**, 1003–1008 (1968).
[4] T. Furuta and Z. Takeda: A characterization of spectraloid operators and its generalization. Proc. Japan Acad., **43**, 599–604 (1967).
[5] J. Holbrook: On the power-bounded operators of Sz-Nagy and Foias. Acta. Sci. Math., **29**, 299–310 (1968).
[6] Sz-Nagy: Sur contractions de l'espace de Hilbert. Acta. Sci. Math., **15**, 87–92 (1953).
[7] Sz-Nagy and C. Foias: On certain classes of power bounded operators in Hilbert space. Acta. Sci. Math., **27**, 17–25 (1966).