122. *Korteweg-deVries Equation. I*

Global Existence of Smooth Solutions

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In this note we state the result only. The detailed proof will be published elsewhere. We treat in this note (I) the global existence of smooth solutions of the Cauchy problem for the KdV equation. In the following note (II) [6] we show the existence of computable approximate solutions by the method of finite difference schemes.

Consider the Cauchy problem for the KdV equation.

(1) $D_t u = uDu + D^3 u + g(x, t) \qquad (x, t) \in R^1 \times (0, \infty)$

(2) $u(x, 0) = f(x) \qquad x \in R^1$

Here $D_t = \dfrac{\partial}{\partial t}$, $D = \dfrac{\partial}{\partial x}$

Main theorem.

If

$$f(x) \in \mathcal{E}_{L^2}^{3(k+1)}(R^1)$$
$$g(x, t) \in \mathcal{E}_t^{k+1}(L^2) \cap [\mathcal{E}_t^k(L^2) \cap \mathcal{E}_t^{k-1}(\mathcal{E}_{L^2}^3) \cap \cdots \cap \mathcal{E}_t^0(\mathcal{E}_{L^2}^{3k})]$$

then there exists uniquely the solution $u(x, t)$ of the Cauchy problem for the KdV equation for $0 \leq t < \infty$ such that

$$u(x, t) \in \mathcal{E}_t^k(L^2) \cap \mathcal{E}_t^{j-1}(\mathcal{E}_{L^2}^3) \cap \cdots \cap \mathcal{E}_t^0(\mathcal{E}_{L^2}^{3k}).$$

Using Sobolev's lemma we conclude easily following corollaries

Corollary 1.

If

$$f(x) \in \mathcal{E}_{L^2}^{3(k+3)}(R^1)$$
$$g(x, t) \in \mathcal{E}_t^{k+3}(L^2) \cap [\mathcal{E}_t^{k+2}(L^2) \cap \cdots \cap \mathcal{E}_t^0(\mathcal{E}_{L^2}^{3(k+2)})]$$

then for any i, j such that $i + j \leq k$

$$D_t^i D^{3j} u \in \mathcal{B}^0(R^1 \times [0, T]) \quad \textit{for } \forall T > 0.$$

Remark. In Corollary 1 if we take $k=1$ we obtain the global existence theorem of classical solutions of KdV equation.

Corollary 2.

If

$$f(x) \in \mathcal{E}_{L^2}^{\infty}$$
$$g(x, t) \in \mathcal{E}_t^{\infty}(\mathcal{E}_{L^2}^{\infty})$$

then

$$u(x, t) \in \mathcal{E}_t^{\infty}(\mathcal{E}_{L^2}^{\infty})$$

especially

$$u(x, t) \in \mathcal{B}^{\infty}(R^1 \times [0, T]) \quad \textit{for } \forall T > 0.$$

To prove the global existence theorem we need the global a priori estimate and the local existence theorem. For the a priori estimate we use the results of Miura-Gardner-Kruskal [2] who showed the existence of infinite sequence of conserved densities for KdV equation. We state their results in a slightly modified form

Theorem 1. *For any non-negative integer k, KdV equation has the polynomial conserved density of the form*

$$T_k=(D^k u)^2+c_k u(D^{k-1}u)^2+Q_k(u, \cdots, D^{k-2}u)$$
$$T_0=u^2.$$

Here c_k is a constant independent of u, Q_k is a polynomial of rank $k+2$.

Definition 1. T is called polynomial conserved density if and only if T is a polynomial in finite number of $D^k u$'s $(k=0, 1, 2, \cdots)$ and there exist X which is also polynomial in $D^k u$'s such that $D_t T=DX$.

Definition 2. A polynomial Q is called rank m if Q is a sum of finite number of monomials of same rank m. For a monomial we define

$$\text{rank } [u^{\alpha_0}(Du)^{\alpha_1}\cdots(D^l u)^{\alpha_l}]=\sum_{j=0}^{l}\frac{1}{2}(j+2)\alpha_j.$$

By integrating these polynomial conserved densities on the whole x-axis we get step by step following infinite number of a priori estimate for the solutions of the KdV equation.

Theorem 2. *For any non-negative integer k, the solutions of the KdV equation satisfy a priori estimate of the form*

$$|||D^k u|||_t^2\le F_k(||f||, \cdots, ||D^k f||)+G_k(t, |||g|||_t, \cdots, |||D^k g|||_t).$$

Here

$$|||u|||_t=\sup_{0\le s\le t}||u(s)||, \quad ||u||=||u||_{L^2(R^1)}$$

F_k, G_k are polynomials of positive coefficients.

Next we show the local existence theorem by constructing the sequence of approximate solutions. Namely we consider the linearlized equations of the KdV equation

$$(3) \qquad u_0(x, t)=f(x)$$

$$(4) \qquad \begin{cases} D_t u_n=u_{n-1}Du_n+D^3u_n+g(x, t) \\ u_n(x, 0)=f(x) \qquad n=1, 2, 3, \cdots. \end{cases}$$

By induction in n and k we obtain following uniform local energy estimates for approximate solutions.

Proposition 1. *For any non-negative integer k, if we take*

$$t_k=\min\left\{\frac{1}{C_{k,0}}, \frac{1}{C_1}, 1\right\}$$

then for any i, j such that $i+j\le k$ there exist uniform energy estimate for approximate solutions of the form

$$|||D_t^i D^{3j}u_n|||_{t_k}\le c_{i,j} \qquad n=0, 1, 2, \cdots.$$

Here we use following notations

$$|||g|||_{k,t}=\sum_{i+j\le k}|||D_t^iD^{3j}g|||_t$$

$$\begin{cases} u(0)=f(x) \\ u^{(1)}(0)=f(x)Df(x)+D^3f(x)+g(x,0) \\ u^{(k)}(0)=\sum_{i=0}^{k-1}\binom{k-1}{i}u^{(k-1-i)}(0)Du^{(i)}(0)+D^3u^{(k-1)}(0)+D_t^{k-1}g(x,0) \end{cases}$$

$$c_{k,0}=M[\|u^{(k)}(0)\|+\cdots+\|u(0)\|+|||D_t^kg|||_t+|||g|||_{k-1,t}]$$

$$c_{k-l,l}=M[c\|u^{(k)}(0)\|+C_{k,0}\{\|u^{(k-1)}(0)\|+\cdots+\|u(0)\|\}+c|||D_t^kg|||_t+|||g|||_{k-1,t}]+C_{k-1}|||g|||_{k-1,t}$$

$$l=1,2,\cdots,k.$$

Here c is a constant independent of f, and g. M is a fixed constant such that $M>1$. C_{k-1} is a polynomial of $c_{i,j}$ for $i+j\le k-1$ with positive coefficients. $C_{k,0}$ is a polynomial of $c_{i,j}$ for $i+j\le k-1$ and $c_{k,0}$ with positive coefficients.

From (4) we derive the following equations for the differences $u_{n+1}-u_n=\varphi_n$

$$(5)\qquad \begin{cases} D_t\varphi_n=u_nD\varphi_n+\varphi_{n-1}Du_n+D^3\varphi_n \\ \varphi_n(x,0)=0 \qquad n=0,1,2,\cdots. \end{cases}$$

Using uniform estimate for u_n in Proposition 1 we obtain following estimate for φ_n.

Proposition 2. *For any non-negative integer k, if we take $T_{k+1}=\rho/C_{k+1}$ here ρ is a fixed constant such that $0<\rho<1$ there exists following estimate*

$$[|||\varphi_n|||_{k,T_{k+1}}+|||\varphi_{n-1}|||_{k-1,T_{k+1}}+\cdots+|||\varphi_{n-k}|||_{0,T_{k+1}}]$$
$$\le\rho[|||\varphi_{n-1}|||_{k,T_{k+1}}+|||\varphi_{n-2}|||_{k-1,T_{k+1}}+\cdots+|||\varphi_{n-k-1}|||_{0,T_{k+1}}].$$

From this estimate it follows easily that

$$D_t^iD^{3j}u_n\to D_t^iD^{3j}u \text{ in } \mathcal{E}_t^0(L^2) \text{ as } n\to\infty \text{ for } i+j\le k.$$

Therefore we obtain following local existence theorem.

Theorem 3. *If*

$$f(x)\in\mathcal{E}_{L^2}^{3(k+1)}$$
$$g(x,t)\in\mathcal{E}_t^{k+1}(L^2)\cap[\mathcal{E}_t^k(L^2)\cap\mathcal{E}_t^{k-1}(\mathcal{E}_{L^2}^3)\cap\cdots\cap\mathcal{E}_t^0(\mathcal{E}_{L^2}^{3k})]$$

then Cauchy problem for the KdV equation has unique solution $u(x,t)$ in $0\le t\le T_{k+1}$ such that

$$u(x,t)\in\mathcal{E}_t^k(L^2)\cap\mathcal{E}_t^{k-1}(\mathcal{E}_{L^2}^3)\cap\cdots\cap\mathcal{E}_t^0(\mathcal{E}_{L^2}^{3k})$$
$$|||u|||_{k+1,T_{k+1}}\le const.$$

Combining this local existence theorem with the global a priori estimate (Theorem 2), we can easily conclude the global existence theorem (Main theorem).

Remark. Uniqueness of the solutions is easily obtained from L^2-energy estimate.

Remark. Our method is also applicable for the Cauchy problem for the KdV equation with the periodic boundary condition. The same results hold in this case (Sjöberg [3] showed the global existence of the classical solutions in this situation).

Remark. Miura [1] discovered the non-linear transformation between the solutions v of the generalized KdV equation $D_t v + v^2 Dv + D^3 v = 0$ and the solutions u of the KdV equation $D_t u + uDu + D^3 u = 0$ such as $u = v^2 \pm i\sqrt{6}\, Dv$.

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