120. On a Subclass of M-Spaces

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1. Introduction. In the present paper all spaces are Hausdorff. In a previous paper [3], K. Morita defined M-space, which is an important generalization of metric and compact spaces. A space X is an M-space iff there is a normal sequence $\{\mathcal{U}_i : i=1, 2. \cdots\}$ of open covers of X satisfying condition (M_0) below;

$$(M_0)\quad \begin{cases} \text{If } \{x_i\} \text{ is a sequence of points in } X \text{ such that} \\ x_i \in \mathrm{St}(x_0, \mathcal{U}_i) \text{ for all } i \text{ and for fixed } x_0 \text{ in } X, \\ \text{then } \{x_i\} \text{ has a cluster point.} \end{cases}$$

Unfortunately, the product of M-spaces may not be M, for which reason T. Ishii, M. Tsuda and S. Kunugi [2] have defined a class $\mathfrak{C}$ of spaces. A space X is of class $\mathfrak{C}$ iff there is a normal sequence $\{\mathcal{U}_i : i=1, 2, \cdots\}$ of open covers of X satisfying condition (*) below:

$$(*)\quad \begin{cases} \text{If } \{x_i\} \text{ is a sequence of points of } X \text{ such that} \\ x_i \in \mathrm{St}(x_0, \mathcal{U}_i) \text{ for all } i \text{ and for fixed } x_0 \text{ in } X, \\ \text{then there is a subsequence } \{x_{i(n)}\} \text{ which has} \\ \text{compact closure.} \end{cases}$$

Ishii, Tsuda and Kunugi have proved in [2] that if a space X is of class $\mathfrak{C}$, then $X \times Y$ is M for any M-space Y; and that the product of countably many spaces of class $\mathfrak{C}$ is also of class $\mathfrak{C}$. They also prove that among the M-spaces belonging to class $\mathfrak{C}$ are:

(a) first countable spaces,
(b) locally compact spaces,
(c) paracompact spaces.

The purpose of this paper is to introduce weakly-k spaces (which contain (a) and (b) above) and weakly para-k spaces (which contain (a), (b), and (c) above), in order to improve Ishii, Tsuda and Kunugi's result as follows:

Theorem 1.1. *Given a space X, the following are equivalent:*

(a) *X is of class $\mathfrak{C}$.*
(b) *X is M and weakly-k.*
(c) *X is M and weakly para-k.*

The spaces are defined as follows:

Definition 1.2. X is *weakly-k* iff: given $F \subseteq X$, $F \cap C$ is finite for all C compact in X implies F closed.

Definition 1.3. X is *weakly para-k* iff: given $F \subseteq X$, F has finite intersection with any paracompact closed set $P \subseteq X$ implies F closed.

2. Weakly-k and weakly para-k spaces.

Proposition 2.1. *If X is of class $\mathfrak{C}$, then X is M and weakly-k.*

Proof. Since Ishii, Tsuda and Kunugi [2] proved that any space of class $\mathfrak{C}$ is also M, it suffices to show that X is weakly-k. So assume that $\{\mathcal{U}_i\}$ is a normal sequence in X satisfying (*), and let F be non-closed in X. Take $x \in \mathrm{Cl}F$ such that $x \notin F$. Choose $x_i \in \mathrm{St}(x, \mathcal{U}_i) \cap F$ with $x_i \neq x_j$ for all $i \neq j$. Then $\{x_i\}$ has a subsequence $\{x_{i(n)}\}$ whose closure is compact. Observe that card $(F \cap \mathrm{Cl}\{x_{i(n)}\}) \geqq \aleph_0$. Thus X is weakly-k.

Proposition 2.2. *Any weakly-k space is weakly para-k.*

Proof. In a Hausdorff space any compact subset is closed and paracompact. Hence the result.

Proposition 2.3. *X is of class $\mathfrak{C}$ if X is M and weakly para-k.*

Proof. Let $x_i \in \mathrm{St}(x_0, \mathcal{U}_i)$ for some $x_0 \in X$ and for $\{\mathcal{U}_i\}$ a normal sequence of open covers of X satisfying the (M_0)-property. Take, without loss of generality, $x_i \neq x_j$ for all $i \neq j$. Now $\{x_i\}$ isn't closed (by the (M_0) condition), so there exists a closed paracompact $P \subseteq X$ such that $P \cap \{x_i\} = \{x_{i(n)}\}$ has countably infinite cardinality. Then $\mathrm{Cl}\{x_{i(n)}\} \subseteq P$, since P is closed.

Now $\bigcap_{i=1}^{\infty} \mathrm{St}(x_0, \mathcal{U}_i)$ is countably compact; since, by the (M_0)-property, every sequence $\left\{y_i : y_i \in \bigcap_{i=1}^{\infty} \mathrm{St}(x_0, \mathcal{U}_i)\right\}$ has an accumulation point. Further,

$$C = \{x_i\} \cup \left\{\bigcap_{i=1}^{\infty} \mathrm{St}(x_0, \mathcal{U}_i)\right\}$$

is countably compact, because any subsequence of $\{x_i\}$ has an accumulation point in $\bigcap_{i=1}^{\infty} \mathrm{St}(x_0, \mathcal{U}_i)$ (again by the (M_0)-condition). Since every accumulation point of $\{x_{i(n)}\}$ is in $\bigcap_{i=1}^{\infty} \mathrm{St}(x_0, \mathcal{U}_i)$, $\mathrm{Cl}\{x_{i(n)}\}$ is a closed subset of C. Thus $\mathrm{Cl}\{x_{i(n)}\}$ is countably compact and closed in paracompact P, which implies that $\mathrm{Cl}\{x_{i(n)}\}$ is compact. This last says that X is of class $\mathfrak{C}$.

Combining Propositions 2.1-2.3 above, Theorem 1.1 is obtained.

Franklin [1] has defined sequential spaces:

Definition 2.4. A space X is *sequential* iff: given any $U \subseteq X$, U is open iff every sequence converging to a point $x \in U$ is itself residual (i.e., eventually) in U. Franklin characterized sequential spaces as quotients of metric spaces. He also proved that every first countable space is sequential, and that every sequential space is k. Locally compact spaces are also k, and it is easy to show that every k-space is weakly-k. Thus, by Theorem 1.1,

Corollary 2.5. *Among the M-spaces belonging to class $\mathfrak{C}$ are the sequential spaces and the k-spaces.*

References

[1] S. Franklin: Spaces in which sequences suffice. Fund. Math., **57**, 107–115 (1965).
[2] T. Ishii, M. Tsuda, and S. Kunugi: On the product of M-spaces. I, II. Proc. Japan. Acad., **44**, 897–903 (1968).
[3] K. Morita: Products of normal spaces with metric spaces. Math. Ann., **154**, 365–382 (1964).