160. *On the Dimension of the Product of a Countably Paracompact Normal Space with the Unit Interval*

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(Comm. by Kinjirô KUNUGI, M. J. A., Oct. 13, 1969)

1. Introduction. In 1953, K. Morita [3] proved that $\dim (X\times I) = \dim X+1$ holds if X is a paracompact Hausdorff space, where I denotes the closed unit interval $[0, 1]$ and dim means the covering dimension. He also conjectured that the above equality would be valid if X is countably paracompact normal. In this note we shall answer this problem in the affirmative.

Let us denote by $D(X; G)$ the cohomological dimension of a space X with respect to an abelian group G, that is, $D(X; G)$ is the largest integer n such that $H^n(X, A; G)\neq 0$ for some closed set A of X, where H^* denotes the Čech cohomology based on all locally finite open coverings. We shall prove

Theorem 1. *Let X be a countably paracompact normal space with a finite covering dimension and G a countable abelian group. Then $D(X\times I; G)=D(X; G)+1$.*

As is proved by Y. Kodama [2], the above relation holds for any abelian group G if X is a paracompact Hausdorff space. If we take G=the group of integers Z in Theorem 1, we have $\dim (X\times I) =\dim X+1$, since $D(X; Z)=\dim X$ for each normal space X with a finite covering dimension.

2. Lemmas. The following lemmas are proved in [1].

Lemma 1. *Let X be a countably paracompact normal space and Y a compact metric space. Then the Künneth formula $H^n(X\times Y; G) \cong \sum_{p+q=n} H^p(X; H^q(Y; G))$ holds for each countable abelian group G.*

Lemma 2. *Let X, Y be countably paracompact normal spaces and let A, B be closed sets in X, Y respectively. If $f: (X, A)\to(Y, B)$ is a map such that*

(1) $f|X-A: X-A\to Y-B$ is a onto homeomorphism;

(2) if F is a closed set in X and $F\subset X-A$, then $f(F)$ is closed in Y. Then $f^*: H^*(Y, B; G)\to H^*(X, A; G)$ is a onto isomorphism for each abelian group G.

Let X be a normal space and A a closed set in X. By [2, Lemma 3] for each countable locally finite open covering $\mathfrak{U}$ of A, there exists a countable locally finite open covering $\mathfrak{V}$ of X such that $\mathfrak{V}|A$ is a refinement of $\mathfrak{U}$. Therefore if we denote by $H_c^*(X, A; G)$ the Čech cohomology group of (X, A) with coefficients in G based on all countable locally finite open coverings of X, then H_c^* satisfies the axiom of exactness. On the other hand, if X is countably paracompact normal and G is countable, $H_c^*(X, A; G)$ is naturally isomorphic to $H^*(X, A; G)$ by the proof of [1, Theorem 1]. Hence the cohomology sequence

$$\cdots \to H^n(X, A; G) \to H^n(X; G) \to H^n(A; G) \to H^{n+1}(X, A; G) \to \cdots$$

is exact if (X, A) is a pair of a countably paracompact normal space X and its closed set A and G is a countable abelian group. Let X be a countably paracompact normal space such that $X = X_1 \cup X_2$ with X_i closed in X. Then by Lemma 2, the triad $(X; X_1, X_2)$ is proper and hence the Mayer-Vietoris sequence of $(X; X_1, X_2)$

$$\cdots \to H^n(X; G) \to H^n(X_1; G) \oplus H^n(X_2, G) \to H^n(X_1 \cap X_2; G) \to H^{n+1}(X; G) \to \cdots$$

is exact for each countable abelian group G. Let us denote by $d(X; G)$ the dimension function defined as follows; $d(X; G)$ is the least integer n such that the induced homomorphism $i^*: H^m(X; G) \to H^m(A; G)$ of the inclusion $i: A \subset X$ is epimorphic for each $m \geqq n$ and closed set A in X. As is proved by Skljarenko [4] the equality $D(X; G) = d(X; G)$ holds for each paracompact Hausdorff space X and abelian group G. The proof given there is based on the exactness of the Mayer-Vietoris sequence, and we can prove the following lemma with a slight modification.

Lemma 3. *If X is a countably paracompact normal space, then we have $D(X; G) = d(X; G)$ for each countable abelian group G.*

Corollary. *Under the same assumption of Lemma 3, the following conditions are equivalent:*

(1) $D(X; G) \leqq n$,

(2) *for each $m \geqq n$ and closed set A of X, every map $f: A \to K(G, m)$ is extendable over X, where $K(G, m)$ is a geometrical realization of Eilenberg-Maclane complex as a locally finite simplicial polytope.*

Proof. By [1, Theorem 1] the condition (2) is equivalent to the condition that for each $m \geqq n$ and closed set A of X, the homomorphism $i^*: H^m(X; G) \to H^m(A; G)$ induced by the inclusion map is onto. Thus the equivalence of the conditions (1) and (2) follows from Lemma 3.

The following lemma is a modification of Kodama [2, Theorem 5], but the proof given here seems to be somewhat simpler.

Lemma 4. *Let X be a countably paracompact normal space and Y a compact metric space such that $\dim(X \times Y)$ is finite. Then $D(X \times Y; G)$ is the largest integer n such that $H^n((A_1, A_2) \times (B_1, B_2); G) \neq 0$ for some closed sets $A_1 \subset A_2 \subset X$ and $B_1 \subset B_2 \subset Y$, if G is a countable abelian group.*

Proof. Let us put $D_1(X\times Y;G)=\max\{n|H^n((A_1,A_2)\times(B_1,B_2);G)\neq 0$ for some closed sets $A_1\subset A_2\subset X$ and $B_1\subset B_2\subset Y$. Then, using the exact sequence of triples, it can be seen that $D(X\times Y;G)\geqq D_1(X\times Y;G)$. Suppose that $D(X\times Y;G)>D_1(X\times Y;G)$; we shall prove that this leads a contradiction. Then

(1) $H^n((A_1,A_2)\times(B_1,B_2);G)=0$ for each closed sets $A_1\subset A_2\subset X$ and $B_1\subset B_2\subset Y$, where $n=D(X\times Y;G)$.

On the other hand, there exist a closed set F of $X\times Y$ and a map $f:F\to K(G,n-1)$ which is not extendable over $X\times Y$, by Corollary to Lemma 3. Since $K(G,n-1)$ is a locally finite simplicial polytope and $X\times Y$ is countably paracompact normal, there exists a neighborhood U of F in $X\times Y$ such that f is extendable over U. Then there exist a locally finite open covering $\mathfrak{U}=\{U_\alpha|\alpha\in\Omega\}$ of X and finite open coverings $\mathfrak{W}^\alpha=\{W_i^\alpha|1\leqq i\leqq k(\alpha)\}$, $\alpha\in\Omega$, such that

(2) $\{U_\alpha\times W_i^\alpha|\alpha\in\Omega, 1\leqq i\leqq k(\alpha)$ is a refinement of the covering $\{U, X\times Y-F\}$.

Let $\mathfrak{V}=\{V_\lambda|\lambda\in\Lambda\}$ be a locally finite open covering of X with order $\leqq\dim X+1$, such that $\{\mathrm{St}(x,\mathfrak{V})|x\in X\}$ is a refinement of $\mathfrak{U}$. Let us define for each $\xi=(\lambda_1,\cdots,\lambda_p)\in\Lambda^p$

(3) $F_\xi=X-\cup\{V_\lambda|\lambda\neq\lambda_i, 1\leqq i\leqq p, \lambda\in\Lambda\}$ if $\bigcap_{i=1}^{p}V_{\lambda_i}\neq\emptyset$, and $F_\xi=\emptyset$ if $\bigcap_{i=1}^{p}V_{\lambda_i}=\emptyset$.

Then the covering $\mathfrak{F}=\bigcup_{p=1}^{q}\mathfrak{F}_p$ where $\mathfrak{F}_p=\{F_\xi|\xi\in\Lambda^p\}$ and $q=\dim X+1$, is a locally finite closed refinement of $\mathfrak{U}$. Hence for each $F_\xi\in\mathfrak{F}$, there exists $U_{\alpha(\xi)}\in\mathfrak{U}$, such that $F_\xi\subset U_{\alpha(\xi)}$. Let $\mathfrak{G}^\alpha=\{G_i^\alpha|1\leqq i\leqq k(\alpha)\}$ be a closed refinement of $\mathfrak{W}^\alpha$, for each $\alpha\in\Omega$, such that $G_i^\alpha\subset W_i^\alpha$, $1\leqq i\leqq k(\alpha)$. Then it follows from (2) that $\{F_\xi\times G_i^{\alpha(\xi)}|F_\xi\in\mathfrak{F}, 1\leqq i\leqq k(\alpha(\xi))\}$ is a locally finite closed refinement of $\{U, X\times Y-F\}$. Therefore if we set $F_0=\cup\{F_\xi\times G_i^{\alpha(\xi)}|F\cap(F_\xi\times G_i^{\alpha(\xi)})\neq\emptyset\}$, then $F\subset F_0\subset U$. Since f is extendable over U, there exists a map $f_0:F_0\to K(G,n-1)$ such that $f_0|F=f$. We put $F_p=F_0\cup\{F_\xi\times Y|\xi\in\Lambda^p\}$, $1\leqq p\leqq\dim X+1$, and let us assume that f_0 has an extension $f_{p-1}:F_{p-1}\to K(G,n-1)$ for some $p\geqq 1$. We prove that f_{p-1} can be extended over F_p. Since if $\xi,\xi'\in\Lambda^p$ and $\xi\neq\xi'$, then $F_\xi\cap F_{\xi'}\in\mathfrak{F}_{p-1}$ by (3), it is sufficient to prove that

(4) $f_{p-1,\xi}=f_{p-1}|(F_{p-1}\cap(F_\xi\times Y))$ is extendable over $F_\xi\times Y$ for each $\xi\in\Lambda^p$.

Suppose that $f_{p-1,\xi}$ is extended over $(F_{p-1}\cap(F_\xi\times Y))\cup\cup\{F_\xi\times G_i^{\alpha(\xi)}|1\leqq i\leqq j-1$ for some $j\leqq k(\alpha(\xi))$. Then if we put

(5) $A_1=F_\xi$, $A_2=F_\xi\cap(\cup\{F_{\xi'}|\xi'\in\Lambda^{p-1}\})$, $B_1=G_j^{\alpha(\xi)}$ and $B_2=\cup\{G_j^{\alpha(\xi)}\cap G_i^{\alpha(\xi)}|1\leqq i\leqq j-1\}\cup\cup\{G_j^{\alpha(\xi)}\cap G_i^{\alpha(\xi')}|(F_{\xi'}\times G_i^{\alpha(\xi')})\cap F\neq\emptyset\}$.

Then by (1) $i^*:H^{n-1}(A_1\times B_1;G)\to H^{n-1}((A_1\times B_2)\cup(A_2\times B_1);G)$ induced by the inclusion is epimorphic. Therefore every map of $(A_1\times B_2)$

$\cup(A_2\times B_1)$ is extendable over $A_1\times B_1$ by [1, Theorem 1]. Thus $f_{p-1,\xi}$ is extendable over $(F_{p-1}\cap(F_\xi\times Y))\cup\cup\{F_\xi\times G_i^{\alpha(\xi)}|1\leqq i\leqq j\}$. Hence (4) is proved by the induction. Therefore f_{p-1} extendable to a map f_p of F_p into $K(G, n-1)$. By the induction again, f is extendable over $F_q=X\times Y$, $q=\dim X+1$. But this is a contradiction. Thus the lemma is proved.

3. **Proof of Theorem 1.** Let us put $D(X;G)=n$. Since the theorem is trivial in case $n=0$, we assume that $n>0$. Then there exists a closed set A of X such that $H^n(X, A;G)\neq 0$. If we denote by $X_0=X/A$ the quotient space of X obtained by contracting A into a point a_0. Then we have $H^n(X_0;G)=H^n(X, A;G)$ by Lemma 2. Since $H^{m+1}(X_0\times(I,\dot{I});G)$ is isomorphic to $H^m(X_0;G)$ for each m, $H^{m+1}(X_0\times(I,\dot{I});G)\neq 0$ if $m=n$, and $H^{m+1}(X_0\times(I,\dot{I});G)=0$ if $m>n$. Then the theorem follows from Lemma 4, since $H^*(X_0\times(I,\dot{I});G)\cong H^*((X, A)\times(I,\dot{I});G)$ by Lemma 2.

The following theorem can be proved similarly by Lemma 1 and 4.

Theorem 2. *Let Y be an n-dimensional compact metric space such that $H^n(B_1, B_2;Z)$ contains Z as a direct summand for some closed sets $B_1\subset B_2\subset Y$. Then $D(X\times Y;G)\geqq D(X;G)+n$, for each finite dimensional countably paracompact normal space X and countable abelian group G.*

A compact space C is called a pseudo n-cell if there exists a map f of an n-cell F onto C such that the restriction of f to the boundary of E is a homeomorph (cf. [2]).

Corollary. *Let X and G be as in Theorem 1. If a compact metric space Y contains a pseudo n-cell, then $D(X\times Y;G)\geqq D(X;G)+n$.*

References

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