157. *Mixed Problems for Degenerate Hyperbolic Equations of Second Order*

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1. Introduction. In this note we shall deal with the following equation:

(1.1) $$u_{tt}=p(x)u_{xx}+f(x,t)$$

in $R^1_+\times(0,\infty)$, where $p(x)$ is a real valued function such that;

(i) $p(x)\in C^0(\bar{R}^1_+)$ and $0\leqq p(x)$ ($p(x)$ never vanishes except at $x=0$)

(ii) for $x\to\infty$, $p(x)$ remains bounded, and moreover bounded away from zero

(iii) $p(x)^{-1}$ is summable in the neighborhood of the origin.

Our boundary conditions are as follows:

Case I $u=0$ at $x=0$

Case II $u_x+hu=0$ at $x=0$ (h is a real number).

Since (1.1) is not strictly hyperbolic, we might not expect the mixed problems with above boundary conditions be L^2-well-posed, but we can show that our problem is well suited on a certain function (Hilbert) space.

2. Function spaces $L^2(R^1_+, p^{-1})$ and $H^2(R^1_+, p)$. In this section we establish two function spaces in which we develop our arguments.

Definition 2.1. A distribution $u(x)$ on R^1_+ is said to be in $L^2(R^1_+, p^{-1})$, if and only if

(2.1) $$\|u\|^2_{p^{-1}}=\int_0^\infty |u|^2 p^{-1}dx$$

is finite.

Definition 2.2. A distribution $u(x)$ on R^1_+ is said to be in $H^2(R^1_+, p)$, if and only if

(2.2) $$\|u\|^2_{2,p}=\int_0^\infty (|u|^2+p(x)|u_{xx}|^2)dx$$

is finite.

Lemma 2.3. *If $u(x)$ belongs to $H^2(R^1_+, p)$, then $u_x(0)=\lim u_x(x)$ exists and*

(2.3) $$|u_x(0)|^2\leqslant\varepsilon\int_0^\infty p(x)|u_{xx}|^2\,dx+C(\varepsilon)\int_0^\infty |u|^2\,dx$$

is valid for any positive ε.

Lemma 2.4. *If $u(x)$ is in $H^2(R^1_+, p)$, then $u(x)$ is in $H^1(R^1_+)$ and*

(2.4) $$\int_0^\infty |u_x|^2\,dx\leqslant\varepsilon\int_0^\infty p(x)|u_{xx}|^2\,dx+C(\varepsilon)\int_0^\infty |u|^2\,dx$$

holds for any positive ε.

Lemma 2.5. *Suppose* $u(x)$ *is in* $H^1(R^1_+)$. *We have*

$$\int_0^\infty p^{-1}|u|^2\,dx \leqslant \text{const.}\ \|u\|_1^2. \tag{2.5}$$

Lemma 2.6. *Let* $u(x)$ *be in* $H^2(R^1_+, p)$ *and* $v(x)$ *be in* $H^1(R^1_+)$, *then we obtain the following Green's formula*

$$\int_0^\infty u_{xx}\cdot\bar{v}\,dx = -\int_0^\infty u_x\cdot\bar{v}_x\,dx - u_x(0)\cdot\overline{v(0)}. \tag{2.6}$$

3. Stationary problems. Let us denote

$$D(R^1_+, p)=\{u\in H^2(R^1_+, p)\,;\, u(x)=0 \text{ at } x=0\}$$

and denote

$$N(R^1_+, p)=\{u\in H^2(R^1_+, p)\,;\, u_x(x)+hu(x)=0 \text{ at } x=0\}.$$

Proposition 3.1. *Let* $c(\neq 0)$ *be a real number, then* $-p(x)D_x^2+c^2$ *is a bijection from* $D(R^1_+, p)$ *onto* $L^2(R^1_+, p^{-1})$.

Proof. Suppose $-p(x)u_{xx}+c^2u=0$, then it follows

$$-\int_0^\infty u_{xx}\cdot\bar{u}\,dx + c^2\int_0^\infty p(x)^{-1}|u|^2\,dx=0, \tag{3.1}$$

hence by Lemma 2.6. we have

$$\int_0^\infty |u_x|^2\,dx + c^2\int_0^\infty p(x)^{-1}|u|^2\,dx=0, \tag{3.2}$$

thus u must be identically zero. Now consider the following sesqui-linear form on $H_0^1(R^1_+)$:

$$B[u, v]=(u_x, v_x)+c^2(u, v)_{p^{-1}}, \tag{3.3}$$

where $(\ ,\ )_{p^{-1}}$ denotes the inner product of $L^2(R^1_+, p^{-1})$, then we can see easily, by Lemma 2.5., $B[u, u]$ gives an equivalent norm to the usual one in $H_0^1(R^1_+)$. Thus by the representation theorem of Riesz, we can have a unique element $u(x)$ in $H_0^1(R^1_+)$ such that for any given $g(x)$ in $H_0^{-1}(R^1_+)$

$$B[u, v]=\langle g, \bar{v}\rangle \tag{3.4}$$

holds for any v in $H_0^1(R^1_+)$ and this shows $u(x)$ satisfies as a distribution

$$-u_{xx}+p(x)^{-1}c^2u=g. \tag{3.5}$$

Since we can see $p(x)^{-1}f(x)$ belongs to $H_0^{-1}(R^1_+)$ by Lemma 2.5. if $f(x)$ is in $L^2(R^1_+, p^{-1})$, taking $p(x)^{-1}f(x)$ as $g(x)$ we have

$$-p(x)u_{xx}+c^2u=f \tag{3.6}$$

and finally we can see $u(x)$ is in $D(R^1_+, p)$. This completes the proof.

Proposition 3.2. *If real* β *is large enough in its absolute value, then* $-p(x)D_x^2+\beta^2$ *is a bijective mapping from* $N(R^1_+, p)$ *on to* $L^2(R^1_+, p^{-1})$.

Proof. Suppose $-p(x)u_{xx}+\beta^2u=0$, then we have

$$(u_x, u_x)+\beta^2(u, u)_{p^{-1}}-h|u(0)|^2=0. \tag{3.7}$$

Hence if β^2 is sufficiently large, we obtain $u(x)=0$ identically.

Consider a sesqui-linear from on $H^1(R^1_+)$ given by

$$B_\beta[u, v]=(u_x, v_x)+\beta^2(u, v)_{p^{-1}}-hu(0)\overline{v(0)}, \tag{3.8}$$

then it can be easily seen that $B_\beta[u, u]$ gives an equivalent norm to the usual one in $H^1(R^1_+)$, if β^2 is large enough. Thus for any given g in $H^1(R^1_+)'$, we can find a unique element $u(x)$ in $H^1(R^1_+)$ such that for all $v(x)$ in $H^1(R^1_+)$,

$$B_\beta[u, v] = \langle g, \bar{v} \rangle. \tag{3.9}$$

Taking $p(x)^{-1}f(x)$ as g, where $f(x)$ is in $L^2(R^1_+, p^{-1})$, we have

$$-u_{xx} + \beta^2 p(x)^{-1} u = p(x)^{-1} f(x) \tag{3.10}$$

in the sense of distribution and observing (2.6), we can accomplish the proof.

4. Evolution equation and existence and estimate of solution. We introduce two Hilbert spaces attached to Case I and Case II.

$$\begin{aligned} \mathcal{H}_1 &= H^1_0(R^1_+) \times L^2(R^1_+, p^{-1}) \\ \mathcal{H}_2 &= H^1(R^1_+) \times L^2(R^1_+, p^{-1}) \end{aligned} \tag{4.1}$$

We treat (1.1) as an evolution equation. Set

$$u_1 = u, \quad u_2 = u_t, \tag{4.2}$$

then (1.1) is reduced to

$$D_t U(t) = AU(t) + F(t), \tag{4.3}$$

where $U(t) = {}^t(u_1(t), u_2(t))$, $F(t) = {}^t(0, f(t))$ and

$$A = \begin{bmatrix} 0 & , & 1 \\ p(x)D_x^2, & & 0 \end{bmatrix}. \tag{4.4}$$

According to Case I and Case II, we take the definition domain of A as follows

$$\begin{aligned} D(A)_1 &= D(R^1_+, p) \times H^1_0(R^1_+) \\ D(A)_2 &= N(R^1_+, p) \times H^1(R^1_+). \end{aligned} \tag{4.5}$$

We note $D(A)_1$ and $D(A)_2$ are dense in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively.

Lemma 4.1. *It holds the following estimates*

$$|\mathrm{Re}\,(AU, U)_{\mathcal{H}_j}| \leqslant C_j (U, U)_{\mathcal{H}_j} \tag{4.6}$$

for all U in $D(A)_j$ $(j=1, 2)$.

Proposition 4.2. *If the absolute value of real λ is large enough, then $A-\lambda I$ is a bijective mapping from $D(A)_j$ onto $\mathcal{H}_j$ and it holds with some positive β*

$$\|(A-\lambda I)^{-1}\|_{\mathcal{H}_j} \leqslant (|\lambda| - \beta)^{-1} \quad (|\lambda| > \beta) \ (j=1, 2). \tag{4.7}$$

Thus the direct application of semi-group theory leads us to

Theorem 4.3. *For any initial deta $(u_0(x), u_1(x))$ in $D(A)_j$ and for any $f(x, t)$ in $\mathcal{E}^1_t(L^2(R^1_+, p^{-1}))$, there exists a unique solution $u(x, t)$ of* (1.1) *such that $(u(t), u_t(t), u_{tt}(t))$ is continuous in $H^2(R^1_+, p) \times H^1(R^1_+) \times L^2(R^1_+, p^{-1})$.*

For the energy estimate, we have

Theorem 4.4. *For the solution $u(x, t)$ of* (1.1) *belonging to $\mathcal{E}^0_t(D(A)_j) \cap \mathcal{E}^1_t(H^1(R^1_+)) \cap \mathcal{E}^2_t(L^2(R^1_+, p^{-1}))$, it follows*

$$\|u(t)\|_{2,p} + \|u_t(t)\|_1 + \|u_{tt}(t)\|_{p-1} \leqslant Ce^{\theta t}(\|u_0\|_{2,p} + \|u_1\|_1 + \|f(0)\|_{p-1} + \int_0^t \|f'(s)\|_{p-1}\, ds). \tag{4.8}$$

References

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