156. Some Remarks on Radiation Conditions

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Introduction. In linear wave propagation problems which are time-independent and which take place in unbounded domain, it is in general not possible to characterize the solutions having the desired physical characteristics by imposing only boundedness conditions at infinity. To do so it is necessary to impose sharper conditions at infinity called the radiation conditions.

The radiation conditions have been discovered by Sommerfeld [5] for the reduced acoustic equation (Helmholtz equation) and by Silver [4] and, independently, by Müller [3] for the reduced Maxwell equations (vector Helmholtz equation). On the other hand, in the previous paper [2] the author gave a new formulation of the radiation condition applicable to general hyperbolic systems of Maxwell type, and used it to develop the spectral and scattering theory for the systems in an exterior domain. The acoustic and Maxwell's equations are typical examples of systems of Maxwell type, which suggests that the radiation condition defined in [2] implies both the Sommerfeld and the Silver-Müller radiation conditions. The subject of this note is to verify this by proving that our definition limited to the acoustic (resp. Maxwell's) equation is equivarent to Sommerfeld's (Silver-Müller's) one.

1. A radiation condition for reduced systems of Maxwell type. Let us consider symmetric systems of the form

$$Au=\sum_{j=1}^{n} A_j \frac{\partial u}{\partial x_j}=\lambda u \tag{1}$$

in an exterior domain G of $\boldsymbol{R}^n (n\geq 2)$. Here λ is an arbitrary non-zero complex number, $u=u(x)$ is a $\boldsymbol{C}^m$-valued function of $x=(x_1, x_2, \cdots, x_n) \in G$, and A_j are $m\times m$ Hermitian symmetric matrices with the property

The matrix $A(\xi)=\sum_{j=1}^{n} A_j\xi_j (\xi \in R^n-\{0\})$ *is isotropic, that is,*

$$\det [A(\xi)-\lambda I_m]=\prod_{\nu=1}^{k} (\tau_\nu |\xi| -\lambda)^{m_\nu}, \quad \sum_{\nu=1}^{k} m_\nu=m, \tag{2}$$

where I_m *is the identity in* $\boldsymbol{C}^m$, τ_ν *and* m_ν $(\nu=1, 2, \cdots, k)$ *are constants, and* $|\xi|=(\xi_1^2+\xi_2^2+\cdots+\xi_n^2)^{1/2}$. We say that the operator A is of Maxwell type if the matrix $A(\xi)$ is isotropic (cf., Wilcox [6]).

We lavel $\{\tau_\nu\}$ in decreasing order:

$$\tau_1>\tau_2>\cdots>\tau_k. \tag{3}$$

Then it follows easily that

(4) $\tau_\nu = -\tau_{k-\nu+1}$ and $m_\nu = m_{k-\nu+1}$.

Thus $\tau_\nu \neq 0$ for each ν if k is even, and $\tau_{(k+1)/2}=0$ if k is odd. We denote by $P_\nu(\xi)$ ($\nu=1, 2, \cdots, k$) the projection in C^m onto the eigenspace of $A(\xi)$ corresponding to the eigenvalue $\tau_\nu|\xi|$. $P_\nu(\xi)$ is an $m\times m$ Hermitian symmetric matrix obtained by

$$P_\nu(\xi) = \frac{-1}{2\pi i}\oint_{\Gamma_\nu} [A(\xi)-\lambda I_m]^{-1} d\lambda,$$

where Γ_ν is a small circle about the point $\tau_\nu|\xi|$ containing no other point of $\{\tau_\nu|\xi|\}$. $P_\nu(\xi)$ is a homogeneous function of $\xi \in R^n-\{0\}$ of degree zero and

(5) $P_\nu(-\xi)=P_{k-\nu+1}(\xi)$, $\quad P_{(k+1)/2}(-\xi)=P_{(k+1)/2}(\xi)$.

We define the projections $P_+(\xi)$ and $P_-(\xi)$ in C^m as follows:

(6) $$P_\pm(\xi)=\sum_{\nu=1}^{[k/2]} P_\nu(\pm\xi).$$

Then it follows that

(7) $$P_+(\xi)+P_-(\xi)+P_{(k+1)/2}(\xi)=I_m.$$

Here $P_{(k+1)/2}(\xi)$ is the projection onto the null space of $A(\xi)$, and we have put $P_{(k+1)/2}(\xi)\equiv 0$ if k is even.

Now the radiation condition defined in [2] for the operator A can be stated in the following form:

Definition. *A solution $u=u(x,\lambda)$ of equation* (1) *in an exterior domain G is said to satisfy the radiation condition if it behaves for $|x|$ large like*

$[\mathrm{R.C.1}]_\pm$ $$\begin{cases} u(x,\lambda)=O(|x|^{-(n-1)/2}); \\ \left\{I_m - P_\pm\left(\dfrac{x}{|x|}\right)\right\} u(x,\lambda)=o(|x|^{-(n-1)/2}). \end{cases}$$

The subscripts "+" and "−" denote the "incoming" and "outgoing" radiation conditions, respectively.

$[\mathrm{R.C.1}]_\pm$ is satisfactory from the viewpoint in that it leads to boundary value problems associated with the operator A having unique solutions (see [2]; Theorem 3.1, Lemma 4.1 and Theorem 4.2). In the simplest case, i.e., when $G=R^n$, the uniqueness property can be stated as follows ([2]; Corollary 3.1):

Proposition 1. *Let $f(x)$ be a C^m-valued square integrable function having a bounded support in R^n, and σ be any non-zero real number. Then the solution $u=u_\pm$ of the equation*

(8) $$Au-i\sigma u=f(x) \quad \textit{in } R^n$$

which satisfies $[\mathrm{R.C.1}]_\pm$ *is unique, and is given by*

(9) $$u_\pm(x)=\lim_{\varepsilon\to+0} u(x,\sigma\pm i\varepsilon),$$

where $u(x,\sigma\pm i\varepsilon)$ is the square integrable solution of (8) *with σ replaced by $\sigma\pm i\varepsilon$ ($\varepsilon>0$).*

2. The case of the reduced acoustic equation.

Let $p=p(x)$ satisfy the reduced acoustic equation

$$\nabla^2 p+\mu^2 p=0 \tag{10}$$

in an exterior domain G of $\boldsymbol{R}^3$, where μ^2 is an arbitrary non-zero complex number and μ is the square root of μ^2 which satisfies $\operatorname{Im}\mu\geq 0$. The Sommerfeld radiation condition for p is given as follows:

$$[\text{R.C.2}]_\pm \qquad \begin{cases} p(x)=O(|x|^{-1}) \\ \dfrac{\partial p(x)}{\partial |x|}\pm i\mu p(x)=o(|x|^{-1}) \end{cases} \qquad (\text{as } |x|\to\infty).$$

On the other hand, (10) can be rewritten as a 4×4 matrix system for

$$u=\left(\frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}, \frac{\partial p}{\partial x_3}, i\mu p\right)^*=(\nabla p, i\mu p)^*,$$

where $i=\sqrt{-1}$, and if M is a matrix, M^* denotes the transpose of M. Then the equation has the canonical form

$$Au-\lambda u=0 \tag{11}$$

with $\lambda=i\mu$ and

$$A=\sum_{j=1}^{3} A_j D_j=\begin{bmatrix} 0 & 0 & 0 & D_1 \\ 0 & 0 & 0 & D_2 \\ 0 & 0 & 0 & D_3 \\ D_1 & D_2 & D_3 & 0 \end{bmatrix}\left(D_j=\frac{\partial}{\partial x_j}\right). \tag{12}$$

Lemma 1. *A C^4-valued function $u=u(x)$ satisfies* (11) *with $\lambda=i\mu$ in G if and only if u has the form $u=(\nabla p, i\mu p)^*$ with p satisfying* (10) *in G.*

Proof. It is sufficient to prove the "only if" part. Suppose that $u=(u_1, u_2, u_3, u_4)^*=(\tilde{u}, u_4)^*$ satisfies (11) in G. Then it follows that

$$\nabla u_4=i\mu\tilde{u} \quad \text{and} \quad \nabla\cdot\tilde{u}=i\mu u_4. \tag{13}$$

Substituting the first equation $\tilde{u}=\dfrac{1}{i\mu}\nabla u_4$ into the second equation, we get $\nabla^2 u_4+\mu^2 u_4=0$ in G. Thus, if we put $p=\dfrac{1}{i\mu}u_4$, then $\tilde{u}=\nabla p$ and hence $u=(\nabla p, i\mu p)^*$. q.e.d.

We put

$$A(\xi)=\begin{bmatrix} & & & \xi_1 \\ & 0 & & \xi_2 \\ & & & \xi_3 \\ \xi_1 & \xi_2 & \xi_3 & 0 \end{bmatrix}=\begin{bmatrix} 0 & \xi^* \\ \xi\cdot & 0 \end{bmatrix}. \tag{14}$$

Then, since

$$\det[A(\xi)-\lambda I_4]=\lambda^2(\lambda^2-|\xi|^2)=\prod_{\nu=1}^{3}(\tau_\nu|\xi|-\lambda)^{m_\nu}, \tag{15}$$

where $\tau_1=1$, $\tau_2=0$, $\tau_3=-1$ and $m_1=1$, $m_2=2$, $m_3=1$, the operator A defined by (12) is of Maxwell type.

Lemma 2. $[\mathrm{R.C.1}]_{\pm}$ *for A defined by* (12) *can be represented as follows*:

$$[\mathrm{R.C.1}]_{A,\pm} \quad \begin{cases} u(x)=O(|x|^{-1}); \\ \left\{A\left(\dfrac{x}{|x|}\right)\pm I_4\right\}u(x)=o(|x|^{-1}) \end{cases} \quad (\text{as } |x|\to\infty).$$

Proof. Since the inverse of the matrix $A(\xi)-\lambda I_4$ is obtained as

$$[A(\xi)-\lambda I_4]^{-1}=\frac{1}{\lambda(\lambda^2-\xi^2)}\begin{bmatrix} \xi_2^2+\xi_3^2-\lambda^2 & -\xi_1\xi_2 & -\xi_1\xi_3 & \lambda\xi_1 \\ -\xi_2\xi_1 & \xi_3^2+\xi_1^2-\lambda^2 & -\xi_2\xi_3 & \lambda\xi_2 \\ -\xi_3\xi_1 & -\xi_3\xi_2 & \xi_1^2+\xi_2^2-\lambda^2 & \lambda\xi_3 \\ \lambda\xi_1 & \lambda\xi_2 & \lambda\xi_3 & -\lambda^2 \end{bmatrix},$$

we have easily

$$P_1(\xi)=P_3(-\xi)=\frac{1}{2|\xi|^2}\begin{bmatrix} (\xi\cdot)\xi^* & -|\xi|\xi^* \\ -|\xi|\xi\cdot & |\xi|^2 \end{bmatrix},$$

$$P_2(\xi)=\frac{-1}{|\xi|^2}\begin{bmatrix} [\xi\times(\xi\times)]^* & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that $P_1(\pm\xi)P_2(\xi)=0$. Then the second condition of $[\mathrm{R.C.1}]_{\pm}$ is equivalent to

$$\left\{I_4-P_{\pm}\left(\frac{x}{|x|}\right)\right\}u+P_1\left(\mp\frac{x}{|x|}\right)u=\left\{2P_1\left(\mp\frac{x}{|x|}\right)+P_2\left(\frac{x}{|x|}\right)\right\}u=o(|x|^{-1}).$$

Here

$$2P_1\left(\mp\frac{x}{|x|}\right)+P_2\left(\frac{x}{|x|}\right)=\begin{bmatrix} I_3 & \pm\dfrac{x^*}{|x|} \\ \pm\dfrac{x}{|x|}\cdot & 1 \end{bmatrix}=A\left(\pm\frac{x}{|x|}\right)+I_4$$

since $-\xi\times(\xi\times f)+(\xi\cdot f)\xi=|\xi|^2 f$ for every f in $\boldsymbol{C}^3$. q.e.d.

Now our problem is to prove the following

Theorem 1. $[\mathrm{R.C.2}]_{\pm}$ *and* $[\mathrm{R.C.1}]_{A,\pm}$ *are equivalent.*

Proof. By Lemma 1 we can put $u=(\nabla p, i\mu p)^*$ in $[\mathrm{R.C.1}]_{A,\pm}$:

$$\left\{A\left(\frac{x}{|x|}\right)\pm I_4\right\}u=\begin{bmatrix} i\mu p\dfrac{x^*}{|x|}\pm[\nabla p]^* \\ \dfrac{\partial p}{\partial|x|}\pm i\mu p \end{bmatrix}=o(|x|^{-1}).$$

Thus, in order to show the theorem, we have only to verify that

$$\nabla p=O(|x|^{-1}) \quad \text{and} \quad i\mu p\frac{x}{|x|}\pm\nabla p=o(|x|^{-1})$$

if $p=p(x)$ is assumed to satisfy (10) in G and $[\mathrm{R.C.2}]_{\pm}$, which are evident since $p(x)$ can be represented for $|x|>\rho+1$ (large) as

$$p(x)=\frac{-1}{4\pi}\int_{\rho<|y|<\rho+1}\frac{e^{\mp i\mu|x-y|}}{|x-y|}\{[\nabla^2\alpha(y)]p(y)+\nabla\alpha(y)\cdot\nabla p(y)\}dy$$

by use of the Green formula (cf., e.g., Mizohata [1], Chapter VIII), where $\alpha(x)$ is a C^∞ function which is identically one for $|x|>\rho+1$ and vanishes inside the ball $\{|x|<\rho\}$. q.e.d.

3. The case of the reduced Maxwell equations.

Let $u=(e, m)$ satisfy the reduced Maxwell equations

$$\begin{cases} \nabla\times m - i\mu e = 0 \\ -\nabla\times e - i\mu m = 0 \end{cases} \tag{16}$$

in an exterior domain G of $\boldsymbol{R}^3$, where μ is an arbitrary non-zero complex number which satisfies Im $\mu\geq 0$. The Silver-Müller radiation conditions for $u=(e, m)$ can be written as follows:

$$[\text{R.C.3}]_{\pm} \qquad \begin{cases} e(x)=O(|x|^{-1}); \qquad m(x)=O(|x|^{-1}); \\ \dfrac{x}{|x|}\times(\nabla\times e)\mp i\mu e = o(|x|^{-1}); \\ \dfrac{x}{|x|}\times(\nabla\times m)\mp i\mu m = o(|x|^{-1}), \qquad \text{as } |x|\to\infty. \end{cases}$$

The matrix $A(\xi)$ corresponding to the Maxwell operator $\begin{bmatrix} 0 & \nabla\times \\ -\nabla\times & 0 \end{bmatrix}$ are the following:

$$A(\xi)=\begin{bmatrix} 0 & B(\xi) \\ B(-\xi) & 0 \end{bmatrix}, \quad \text{where} \quad B(\xi)=\begin{bmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{bmatrix}=(\xi\times)^*. \tag{17}$$

Since

$$\det[A(\xi)-\lambda I_6]=\lambda^2(\lambda^2-|\xi|^2)^2=\prod_{\nu=1}^{3}(\tau_\nu|\xi|-\lambda)^2, \tag{18}$$

where $\tau_1=1$, $\tau_2=0$, $\tau_3=-1$, the Maxwell operator is also of Maxwell type.

Lemma 3. $[\text{R.C.1}]_{\pm}$ *for the Maxwell operator can be represented as follows*:

$$[\text{R.C.1}]_{M,\pm} \qquad \begin{cases} u(x)=O(|x|^{-1}); \\ \left\{A\left(\dfrac{x}{|x|}\right)\pm I_6\right\}u(x)=o(|x|^{-1}) \end{cases} \quad (\text{as } |x|\to\infty),$$

where $u=(e, m)^*$ *with* e *and* m *being* $\boldsymbol{C}^3$*-valued functions.*

Proof. The inverse of the matrix $A(\xi)-\lambda I_6$ is obtained as

$$[A(\xi)-\lambda I_6]^{-1}=\frac{1}{\lambda(\lambda^2-|\xi|^2)}\begin{bmatrix} R(\xi,\lambda) & S(\xi,\lambda) \\ S(-\xi,\lambda) & R(\xi,\lambda) \end{bmatrix},$$

where

$$R(\xi,\lambda)=R(-\xi,\lambda)=\begin{bmatrix} \xi_1^2-\lambda^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_2\xi_1 & \xi_2^2-\lambda^2 & \xi_2\xi_3 \\ \xi_3\xi_1 & \xi_3\xi_2 & \xi_3^2-\lambda^2 \end{bmatrix}, \qquad S(\xi,\lambda)=B(-\lambda\xi).$$

From this it is not difficult to see that

$$P_1(\pm\xi)=\frac{-1}{2|\xi|^2}\begin{bmatrix} [\xi\times(\xi\times)]^* & \pm|\xi|(\xi\times)^* \\ \mp|\xi|(\xi\times)^* & [\xi\times(\xi\times)]^* \end{bmatrix},$$

$$P_2(\xi)=\frac{1}{|\xi|^2}\begin{bmatrix} (\xi\cdot)\xi^* & 0 \\ 0 & (\xi\cdot)\xi^* \end{bmatrix}.$$

Thus, by the same reasoning as in the proof of Lemma 2, we can conclude the assertion of the lemma. q.e.d.

For $u=(e, m)$, $[\mathrm{R.C.1}]_{M,\pm}$ can be rewritten as

$$[\mathrm{R.C.3}]'_{\pm} \qquad \begin{cases} e(x)=O(|x|^{-1})\,; & m(x)=O(|x|^{-1})\,; \\ \dfrac{x}{|x|}\times e\mp m=o(|x|^{-1})\,; & \dfrac{x}{|x|}\times m\pm e=o(|x|^{-1}). \end{cases}$$

This is equivalent to $[\mathrm{R.C.3}]_{\pm}$ if $u=(e, m)$ satisfies (16) for $|x|$ large. Thus we have the following

Theorem 2. $[\mathrm{R.C.3}]_{\pm}$ *and* $[\mathrm{R.C.1}]_{M,\pm}$ *are equivalent.*

References

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