# 154. On Strongly Normal Spaces

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A strongly normal space is a countably paracompact, collectionwise normal space (M. Katětov [2]). M. Katětov [2] and V. Šedivá [5] proved independently the following:

**Theorem 1. A** (Katětov, Šedivá). *A normal space $X$ is a strongly normal space if and only if for every locally finite collection $\{F_\lambda \mid \lambda \in \Lambda\}$ of subsets of $X$ there exists a locally finite collection $\{G_\lambda \mid \lambda \in \Lambda\}$ of open subsets of $X$ such that $F_\lambda \subset G_\lambda$ for each $\lambda \in \Lambda$.*

We have, however, no informations on other characterizations of strongly normal spaces. The purpose of this paper is to obtain some characterizations of strongly normal spaces in terms of "coverings" (Theorem 2. A). Furthermore, we shall also obtain similar characterizations of collectionwise normal spaces (Theorem 2. B).

An open covering of a topological space is called an *A-covering* if it has a locally finite (not necessarily open) refinement. Every countable open covering is an $A$-covering. Indeed, for a countable open covering $\mathfrak{U}=\{U_n \mid n=1, 2, \cdots\}$ the collection $\{V_n \mid n=1, 2, \cdots\}$ is a locally finite refinement of $\mathfrak{U}$, where $V_1=U_1$ and $V_n=U_n-\bigcup_{i=1}^{n-1} U_i$ for $n=2, 3, \cdots$.

A collection of subsets of a topological space is called *bounded locally finite,* if there is a positive integer $n$ such that every point of the space has a neighborhood which intersects only at most $n$ elements of the collection. The following theorem is due to [2, Proposition 3.1].

**Theorem 1. B** (Katětov). *A normal space $X$ is a collectionwise normal space if and only if for every bounded locally finite collection $\{F_\lambda \mid \lambda \in \Lambda\}$ of subsets of $X$ there exists a locally finite collection $\{G_\lambda \mid \lambda \in \Lambda\}$ of open subsets of $X$ such that $F_\lambda \subset G_\lambda$ for each $\lambda \in \Lambda$.*

An open covering of a topological space is called a *B-covering* if it has a bounded locally finite refinement. Of course, every $B$-covering is an $A$-covering and every finite open covering is a $B$-covering.

**Theorem 2. A.** *For a topological space $X$ the following conditions are equivalent:*

(a) *$X$ is a strongly normal space.*

(b) *$X$ is a normal space*[1] *and for every locally finite covering*

1) In our terminology a normal space need not be a Hausdorff space.

*$\{F_\lambda \mid \lambda \in \Lambda\}$ of $X$ there exists a locally finite open covering $\{G_\lambda \mid \lambda \in \Lambda\}$ of $X$ such that $F_\lambda \subset G_\lambda$ for each $\lambda \in \Lambda$.*

(c) *$X$ is a normal space and every $A$-covering of $X$ has a locally finite open refinement.*

(d) *For every $A$-covering $\mathfrak{U}$ of $X$ there exists a locally finite cozero covering*[2] *$\mathfrak{V}$ of $X$ such that $\{\bar{V} \mid V \in \mathfrak{V}\}$ refines $\mathfrak{U}$.*

(e) *Every $A$-covering of $X$ has a locally finite closed refinement.*

(f) *Every $A$-covering of $X$ has a closure-pleserving closed refinement.*

(g) *Every $A$-covering of $X$ has a locally finite partition of unity subordinated to it.*

(h) *Every $A$-covering of $X$ has a partition of unity subordinated to it.*

(i) *Every $A$-covering of $X$ is a normal covering* (in the sense of J. W. Tukey [6]).

(j) *Every $A$-covering of $X$ has an open star-refinement.*

(k) *Every $A$-covering of $X$ has a cushioned refinement* (in the sense of E. Michael [3]).

(l) *Every $A$-covering of $X$ has an open $\sigma$-cushioned refinement.*

**Theorem 2. B.** *For a topological space $X$ the following conditions are equivalent:*

(a) *$X$ is a collectionwise normal space.*

(b) *$X$ is a normal space and for every bounded locally finite covering $\{F_\lambda \mid \lambda \in \Lambda\}$ of $X$ there exists a locally finite open covering $\{G_\lambda \mid \lambda \in \Lambda\}$ of $X$ such that $F_\lambda \subset G_\lambda$ for each $\lambda \in \Lambda$.*

(c)—(l) *of Theorem* 2.A *in which $A$-coverings are replaced by $B$-coverings.*

**Remark.** If we replace "$A$-covering" in (d)—(l) of Theorem 2.A by "open covering" (resp. "countable open covering" or "finite open covering"), each of these conditions is equivalent to saying that $X$ is a paracompact normal (resp. countably paracompact normal or normal) space.

We prove only Theorem 2.A. For the proof we shall need Theorem 1.A. The proof of Theorem 2.B is quite analogous to the proof of Theorem 2.A; we have only to use Theorem 1.B instead of Theorem 1.A.

**Proof of Theorem 2. A.** The style of the proof is

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2) A subset $G$ of a topological space $X$ is called a *cozero-set* of $X$ if there exists a real-valued non-negative continuous function $f$ on $X$ such that $G=\{x \mid f(x)>0\}$. A *cozero covering* is a covering all of whose elements are cozero-set.

$$\begin{tikzcd}
 & & & & (e) \arrow[r] & (f) \arrow[drrr] & & & & \\
(a) \arrow[r] & (b) \arrow[r] & (c) \arrow[r] & (d) \arrow[r] \arrow[ur] \arrow[drr] & (g) \arrow[r] & (h) \arrow[r] & (i) \arrow[r] & (j) \arrow[r] & (k) \arrow[r] & (a). \\
 & & & & & (l) \arrow[urrr] & & & &
\end{tikzcd}$$

(a)→(b). This is obvious from Theorem 1.A.

(b)→(c). Let $\mathfrak{U}$ be an $A$-covering of $X$. Then $\mathfrak{U}$ has a locally finite refinement $\{F_\lambda \mid \lambda \in \Lambda\}$. Each $F_\lambda$ is contained in some element of $\mathfrak{U}$; let it be $U_\lambda$. By assumption there exists a locally finite open covering $\{G_\lambda \mid \lambda \in \Lambda\}$ of $X$ such that $F_\lambda \subset G_\lambda$ for each $\lambda \in \Lambda$. Put $V_\lambda = U_\lambda \cap G_\lambda$ for each $\lambda \in \Lambda$. It is obvious that $\{V_\lambda \mid \lambda \in \Lambda\}$ is a locally finite open refinement of $\mathfrak{U}$.

(c)→(d). As is well known, for every locally finite open covering $\{U_\lambda \mid \lambda \in \Lambda\}$ of a normal space there exists an open covering $\{V_\lambda \mid \lambda \in \Lambda\}$ of the space such that $\bar{V}_\lambda \subset U_\lambda$ for each $\lambda \in \Lambda$. Moreover, each $V_\lambda$ may be a cozero-set.

(d)→(e), (e)→(f) and (f)→(k). These are obvious.

(d)→(g). Let $\mathfrak{U}$ be an $A$-covering of $X$, and let $\mathfrak{V} = \{V_\lambda \mid \lambda \in \Lambda\}$ be a locally finite cozero covering of $X$ such that $\{\bar{V}_\lambda \mid \lambda \in \Lambda\}$ refines $\mathfrak{U}$. Then we have, for each $\lambda \in \Lambda$, a real-valued non-negative continuous function $f_\lambda$ on $X$ such that $V_\lambda = \{x \mid f_\lambda(x) > 0\}$. Put $f(x) = \sum_{\lambda \in \Lambda} f_\lambda(x)$ for $x \in X$, then $f$ is a positive continuous function on $X$ because $\mathfrak{V}$ is a locally finite open covering of $X$. If we define $g_\lambda(x) = f_\lambda(x)/f(x)$ for $x \in X$, then $\{g_\lambda \mid \lambda \in \Lambda\}$ is a locally finite partition of unity subordinated to $\mathfrak{U}$.

(g)→(h). This is obvious.

(h)→(i). This is an immediate consequence of K. Morita [4, Theorem 1.2].

(i)→(j). This is obvious.

(j)→(k). An open star-refinement of a covering is a cushioned refinement of the covering.

(d)→(l). This is obvious.

(l)→(k). As is shown in the proof of [3, Theorem 1.2], a covering with an open $\sigma$-cushioned refinement has a cushioned refinement.

(k)→(a). First we shall prove that $X$ is collectionwise normal. Let $\{F_\lambda \mid \lambda \in \Lambda\}$ be a discrete collection of closed subsets of $X$; we must find a mutually disjoint collection $\{G_\lambda \mid \lambda \in \Lambda\}$ of open subsets of $X$. Put $U_\lambda = X - \bigcup_{\mu \neq \lambda} F_\mu$ for each $\lambda \in \Lambda$, then $U_\lambda$ is open and $U_\lambda \supset F_\lambda$ for each $\lambda \in \Lambda$. Obviously, the collection $\{F_\lambda \mid \lambda \in \Lambda\} \cup \{X - \bigcup_{\lambda \in \Lambda} F_\lambda\}$ is a (bounded) locally finite covering of $X$ which refines $\mathfrak{U} = \{U_\lambda \mid \lambda \in \Lambda\}$. Thus $\mathfrak{U}$ is an $A$-covering (more precisely, a $B$-covering) of $X$ and hence, by assumption, $\mathfrak{U}$ has a cushioned refinement. By [3, Proposition 2.1], we can take a covering $\{V_\lambda \mid \lambda \in \Lambda\}$ such that for every subset $M$ of

$\Lambda$ $\overline{\bigcup_{\lambda \in M} V_\lambda} \subset \bigcup_{\lambda \in M} U_\lambda$. Put $G_\lambda = X - \overline{\bigcup_{\mu \neq \lambda} V_\mu}$ for each $\lambda \in \Lambda$, then each $G_\lambda$ is open. Obviously, the collection $\{G_\lambda \mid \lambda \in \Lambda\}$ is mutually disjoint. For each $\lambda \in \Lambda$,

$$G_\lambda = X - \overline{\bigcup_{\mu \neq \lambda} V_\mu} \supset X - \bigcup_{\mu \neq \lambda} U_\mu = \bigcap_{\mu \neq \lambda} (X - U_\mu) = \bigcap_{\mu \neq \lambda} \bigcup_{\nu \neq \mu} F_\nu \supset F_\lambda.$$

Therefore $X$ is collectionwise normal.

Next, we shall prove that $X$ is countably paracompact. Let $\mathfrak{U} = \{U_n \mid n = 1, 2, \cdots\}$ be a countable open covering of $X$. Since every countable open covering is an $A$-covering, $\mathfrak{U}$ is an $A$-covering of $X$ and hence $\mathfrak{U}$ has a cushioned refinement $\{V_n \mid n = 1, 2, \cdots\}$ such that $\bar{V}_n \subset U_n$ for $n = 1, 2, \cdots$. Thus $X$ is countably paracompact by C. H. Dowker [1, Theorem 2].

## References


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