153. *On Mixed Problems for First Order Hyperbolic Systems with Constant Coefficients*

By Takashi SADAMATSU

(Comm. by Kinjirô KUNUGI, M. J. A., Oct. 13, 1969)

1. Introduction. Mixed problems for linear hyperbolic equations with constant coefficients in a quarter space has been treated by S. Agmon [1], R. Hersh [2] and L. Sarason [6].

In this note, we consider the mixed problem for first order hyperbolic systems with the principal part

$$(1.1)\qquad \begin{cases} L[u]\equiv \dfrac{\partial}{\partial t}u+A\dfrac{\partial}{\partial x}u+\sum\limits_{j=1}^{n}B_j\dfrac{\partial}{\partial y_j}u=f(t;x,y)\\ u(0;x,y)=0\\ Pu(t;0,y)=0 \end{cases}$$

in the quarter space $\{(t;x,y);\ t>0,\ x>0,\ y\in R^n\}$, where u is a N-vector, A, $B_j(j=1,2,\cdots,n)$ $N\times N$-constant matrices and P $m\times N$-constant matrix of rank m. A is supposed to be non-singular.

Our argument is based on Wiener-Hopf's method. After Laplace transformation in t and Fourier transformation in y, the problem (1.1) is translated into the following equation

$$(1.2)\qquad \begin{cases} \left(A\dfrac{d}{dx}+\tau I+i\sum\limits_{j=1}^{n}\eta_jB_j\right)\hat{u}(\tau;x,\eta)=\hat{f}(\tau;x,\eta)\\ P\hat{u}(\tau;0,\eta)=0, \end{cases}$$

where $\hat{u}(\tau;x,\eta)$ denotes the Fourier-Laplace image of $u(t;x,y)$. Using a compensating function $\hat{g}(\tau;x,\eta)$ which shall be constructed later and setting $u=v+w$, we decompose the problem (1.2) to two problems

$$(1.3)\qquad \left(A\frac{d}{dx}+\tau I+i\sum_{j=1}^{n}\eta_jB_j\right)\hat{v}(\tau;x,\eta)=\hat{f}(\tau;x,\eta)+\hat{g}(\tau;x,\eta)$$

in $x\in R^1$ and

$$(1.4)\qquad \begin{cases} \left(\dfrac{d}{dx}+M(\tau,\eta)\right)\hat{w}(\tau;x,\eta)=0\\ P\hat{w}(\tau;0,\eta)=-P\hat{v}(\tau;0,\eta) \end{cases}$$

where $M(\tau,\eta)=A^{-1}\left(\tau I+i\sum_{j=1}^{n}\eta_jB_j\right)$. Thus we are to treat the problems (1.3) and (1.4).

2. Assumptions and result. *Condition* I. The operator L is hyperbolic in the following sense: 1) the matrix $\xi A+\eta B$ $\left(\eta B\text{ stands for }\sum_{j=1}^{n}\eta_jB_j\right)$ has only real eigenvalues for any real (ξ,η), 2) the matrix

$\xi A+\eta B$ is diagonalizable and the multiplicities of eigenvalues are constant for any real $(\xi, \eta)\neq(0, 0)$, i.e. we have

$$\det(\tau I+i\xi A+i\eta B)=\prod_{j=1}^{s}(\tau-i\lambda_j(\xi, \eta))^{p_j} \tag{2.1}$$

with $\lambda_i(\xi, \eta)$ $(i=1, 2, \cdots, s)$ real and distinct for any real $(\xi, \eta)\neq(0, 0)$ and $p_j(j=1, 2, \cdots, s)$ constants $(p_1+p_2+\cdots+p_s=N)$.

Condition II. For any real η and for any pure imaginary $\tau(=i\gamma$; γ: real), the real roots in ξ of $\det(\tau I+i\xi A+i\eta B)=0$ are at most double in the sense of the remark below for any real $(\gamma, \eta)\neq(0, 0)$.

Remark. Let $\tau=\tau^0=i\gamma^0$ (γ^0: real), $\eta=\eta^0$ and ξ^0 be a real double root of $\det(\tau^0 I+i\xi A+i\eta^0 B)=0$. Then a real double root means that $\frac{\partial}{\partial\xi}\lambda_i(\xi^0, \eta^0)=0$ and $\frac{\partial^2}{\partial\xi^2}\lambda_i(\xi^0, \eta^0)\neq 0$. A real simple root means

$$\frac{\partial}{\partial\xi}\lambda_i(\xi^0, \eta^0)\neq 0.$$

Let $E^+(\tau, \eta)$ (resp. $E^-(\tau, \eta)$) be the subspace of C^N generated by the ordinary and the generalized eigenvectors corresponding to the roots in ξ of $\det(i\xi I+M(\tau, \eta))=0$ with positive (resp. negative) imaginary parts when $\operatorname{Re}\tau>0$. From Conditions I and II, we can construct at least locally a system of vectors $\{h_j^+(\tau, \eta)\}_{j=1,2,\cdots,m}$ continuous and homogeneous of degree zero in τ and η which is a base of $E^+(\tau, \eta)$ when $\operatorname{Re}\tau>0$ and remains linearly independent still when $\operatorname{Re}\tau\geq 0$ (see, M. Mizohata [4], M. Matsumura [3]).

Condition III. The absolute value of Lopatinski determinant is uniformly bounded away from 0 in $|\tau|^2+|\eta|^2=1$ $(\operatorname{Re}\tau\geq 0)$, that is, there exists a positive constant δ such that

$$|\det P\mathcal{H}(\tau, \eta)|\geq\delta \qquad \text{for} \qquad |\tau|^2+|\eta|^2=1 \quad \operatorname{Re}\tau\geq 0.$$

holds, where $\mathcal{H}(\tau, \eta)$ is a $N\times m$-matrix $(h_1^+(\tau, \eta), \cdots, h_m^+(\tau, \eta))$. Then we have

Theorem. *Under Conditions* I, II *and* III, *we have the inequality*

$$\|\hat{u}(\tau\,;x, \eta)\|_{L^2(R^1_+)}\leq\frac{\text{const.}}{\operatorname{Re}\tau}\|\hat{f}(\tau\,;x, \eta)\|_{L^2(R^1_+)}$$

for any solution $\hat{u}(\tau\,;x, \eta)$ of the problem (1.2) *where the constant does not depend on τ and η.*

3. Sketch of the proof. In this section we treat the problems (1.3) and (1.4) and give a sketchy proof of the theorem assuming some lemmas. The solution $\hat{v}(\tau\,;x, \eta)$ in $L^2(R^1)$ of the problem (1.3) can be represented by

(3.1)

$$\hat{v}(\tau\,;x, \eta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ix\cdot\xi}(\tau I+i\xi A+i\eta B)^{-1}\{\tilde{\hat{f}}(\tau\,;\xi, \eta)+\tilde{\hat{g}}(\tau\,;\xi, \eta)\}d\xi$$

where $\tilde{\hat{f}}(\tau; \xi, \eta)$ (briefly $\tilde{f}(\xi)$) denotes Fourier image of $\hat{f}(\tau; x, \eta)$ in x and

$$(3.2) \qquad P\hat{v}(\tau; 0, \eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(\tau I + i\xi A + i\eta B)^{-1}\{\tilde{f}(\xi) + \tilde{g}(\xi)\}d\xi$$

Lemma 1. *Under Condition* I, *the inequality*

$$(3.3) \qquad |(\tau I + i\xi A + i\eta B)^{-1}| \leq \frac{\text{const.}}{\text{Re}\,\tau}$$

holds for $\text{Re}\,\tau > 0$, *where the constant does not depend on* τ, ξ *and* η.

From (3.1) and Lemma 1, we have the following:

Proposition 1. *Under Condition* I, *the inequality*

$$(3.4) \qquad \|\hat{v}(\tau; x, \eta)\|_{L^2(R^1)} \leq \frac{\text{const.}}{\text{Re}\,\tau} \|\hat{f}(\tau; x, \eta) + \hat{g}(\tau; x, \eta)\|_{L^2(R^1)}$$

holds for the solution $\hat{v}(\tau; x, \eta)$ *of the problem* (1.3).

Lemma 2. *From Condition* I, *the roots in* ξ *of* $\det(\tau I + i\xi A + i\eta B)$ *are never real for any* τ ($\text{Re}\,\tau > 0$) *and real* ξ.

This lemma shows that the numbers of the roots in ξ of $\det(\tau I + i\xi A + i\eta B) = 0$ with positive and negative imaginary parts do not change for any τ ($\text{Re}\,\tau > 0$) and real η.

$$(3.2)' \quad P\hat{v}(\tau; 0, \eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(\tau' I + i\xi' A + i\eta' B)^{-1}\{\tilde{f}(c\xi') + \tilde{g}(c\xi')\}d\xi'$$

where $(\tau', \xi', \eta') = \frac{1}{c}(\tau, \xi, \eta)$ and $c = (|\tau|^2 + |\eta|^2)^{\frac{1}{2}}$. Here we decompose $\det(\tau' I + i\xi' A + i\eta' B)$ into factors:

$$(3.5) \quad \det(\tau' I + i\xi' A + i\eta' B) = i^N \det A \cdot A^+(\xi'; \tau', \eta') A^-(\xi'; \tau', \eta')$$

$$(3.6) \qquad A^+(\xi'; \tau', \eta') = \prod_{j=1}^{m} (\xi' - \xi_j^+(\tau', \tau'))$$

$$(3.7) \qquad A^-(\xi'; \tau', \eta') = \prod_{j=1}^{N-m} (\xi' - \xi_j^-(\tau', \eta'))$$

where $\xi_j^+(\tau', \eta')$ and $\xi_j^-(\tau', \eta')$ are the roots in ξ' of $\det(\tau' I + i\xi' A + i\eta' B) = 0$ with positive and negative imaginary parts respectively. Let us $\tau' = i\gamma^{0'}$, $\eta' = \eta^{0'}$ and suppose that $M(i\gamma^{0'}, \eta^{0'})$ admits a pure imaginary root $i\xi^{0'}$ and that $\gamma^{0'} = \lambda_1(\xi^{0'}, \eta^{0'})$, then we have the following:

Lemma 3. *If we suppose Conditions* I *and* II, *then the rank of* $\tau' I + i\xi' A + i\eta' B$ *is* $N - p_1$ *in a small neighbourhood of* $(\tau^{0'}, \xi^{0'}, \eta^{0'}) = (i\lambda_1(\xi^{0'}, \eta^{0'}), \xi^{0'}, \eta^{0'})$ $(|\tau'|^2 + |\eta'|^2 = 1)$ *when* (τ', ξ', η') *satisfies* $\det(\tau' I + i\xi' A + i\eta' B) = 0$.

With the help of this lemma, we can define the matrix $\mathcal{P}(\xi'; \tau', \eta')$ by

$$(3.8) \qquad P(\tau' I + i\xi' A + i\eta' B)^{-1} = \frac{\mathcal{P}(\xi'; \tau', \eta')}{A_0^+(\xi'; \tau', \eta') A_0^-(\xi'; \tau', \eta')}$$

where

$$A_0^+(\xi'; \tau', \eta') = \prod_{j=1}^{m'} (\xi - \xi_j^+(\tau', \eta')), \ A_0^-(\xi'; \tau', \eta') = \prod_{j=1}^{m''} (\xi' - \xi_j^-(\tau', \eta'))$$

here we changed the notation in the following way: we denotes $\xi_1^+=\cdots=\xi_{p_1}^+$ simply by ξ_1^+, $\xi_{p_1+1}=\cdots$ by ξ_2^+ and so on and $\xi_1^+,\cdots,\xi_p^+$ are all distinct roots of $\det(\tau' I+i\xi' A+i\eta' B)=0$ which approach real roots when (τ',η') tends to $(\tau^{0'}=i\gamma^{0'},\gamma^{0'})$. Further we can decompose

$$\frac{\mathcal{P}(\xi';\tau',\eta')}{A_0^+(\xi';\tau',\eta')A_0^-(\xi';\tau',\eta')}=\frac{\mathcal{P}^+(\xi';\tau',\eta')}{A_0^+(\xi';\tau',\eta')}+\frac{\mathcal{P}^-(\xi';\tau',\eta')}{A_0^-(\xi';\tau',\eta')} \tag{3.9}$$

$$\frac{\mathcal{P}^+(\xi';\tau',\eta')}{A_0^+(\xi';\tau',\eta')}=\sum_{j=1}^{q+s}\frac{c_j^+(\xi',\eta')\mathcal{P}(\xi_j^+;\tau',\eta')}{\xi-\xi_j^+}+R^+(\xi';\tau',\eta') \tag{3.10}$$

$$\frac{\mathcal{P}^-(\xi';\tau',\eta')}{A_0^-(\xi';\tau',\eta')}=\sum_{j=1}^{q+s'}\frac{c_j^-(\tau',\eta')\mathcal{P}(\xi_j^-;\tau',\eta')}{\xi-\xi_j^-}+R^-(\xi';\tau',\eta') \tag{3.11}$$

where $\xi_j^\pm(\tau',\eta')$ $(j=1,2,\cdots,q)$ denote the roots which approach the real double roots $\xi_j^0(i\gamma^{0'},\eta^{0'})$ and $\xi_j^+(\tau',\eta')$ $(j=q+1,\cdots,q+s=p)$, $\xi_j^-(\tau',\eta')$ $(j=q+1,\cdots,q+s')$ denote the roots which approach the real simple roots when (τ',η') tends to $(i\gamma^{0'},\eta^{0'})$.

Lemma 4. *Under Condition* II, *we have*

1) $|c_j^\pm(\tau',\eta')|=0\left(\dfrac{1}{|\xi_j^+-\xi_j^-|}\right)$ *for* $j=1,2,\cdots,q$

2) $\left|\dfrac{c_j^-(\tau',\eta')}{c_j^+(\tau',\eta')}\right|\le\text{const.}$ *for* $j=1,2,\cdots,q$

3) $|c_j^+(\tau',\eta')|\le\text{const.}$ *for* $j=q+1,\cdots,q+s$
$|c_j^-(\tau',\eta')|\le\text{const.}$ *for* $j=q+1,\cdots,q+s'$

4) $|R^\pm(\xi';\tau',\eta')|\le\dfrac{\text{const.}}{1+|\xi|}$ *for real* ξ

for any (τ',η') *in* $V'\cap\{\text{Re}\,\tau'>0\}$ *where* $V'=\dfrac{1}{c}V$ *and* V *is a small neighbourhood of* $(i\gamma^0,\eta^0)$

Lemma 5. *Let* α *and* β *be not real, then the equality*

$$\int_{-\infty}^{\infty}\frac{1}{\xi'-\alpha}\cdot\overline{\frac{1}{\xi'-\beta}}\,d\xi'=\begin{cases}2\pi i\dfrac{1}{\alpha-\overline{\beta}} & \text{for } \text{Im}[\alpha]>0,\ \text{Im}[\beta]>0\\ -2\pi i\dfrac{1}{\alpha-\overline{\beta}} & \text{for } \text{Im}[\alpha]<0,\ \text{Im}[\beta]<0\\ 0 & \text{for } \text{Im}[\alpha]\cdot\text{Im}[\beta]<0\end{cases} \tag{3.12}$$

holds.

Lemma 6. *Under Condition* I, *we have*

$$|\text{Im}\,\xi'(\tau',\eta')|\ge\text{const.}\ \text{Re}\,\tau' \tag{3.13}$$

where $\xi'(\tau',\eta')$ *is a root of* $\det(\tau' I+i\xi' A+i\eta' B)=0$ *in* ξ'.

Lemma 7. *Under Conditions* I *and* II, *we have*

$$\left|\frac{\text{Im}[\xi_j^+(\tau',\eta')]}{\text{Im}[\xi_j^-(\tau',\eta')]}\right|\le\text{const.}\qquad(j=1,2,\cdots,q) \tag{3.14}$$

for (τ',η') *in* $V'\cap\{\text{Re}\,\tau'>0\}$.

Using above decompositions

$$P\hat{v}(\tau\,;0,\eta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{q}\left\{\frac{c_j^+(\tau',\eta')\mathcal{P}(\xi_j^+)}{\xi'-\xi_j^+}\tilde{g}(c\xi')+\frac{c_j^-(\tau',\eta')\mathcal{P}(\xi_j^+)}{\xi'-\xi_j^-}\tilde{f}(c\xi')\right\}d\xi'$$
$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{q}\frac{c_j^-(\tau',\eta')\{\mathcal{P}(\xi_j^-)-\mathcal{P}(\xi_j^+)\}}{\xi'-\xi_j^-}\tilde{f}(c\xi')d\xi' \tag{3.2)'$$
$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=q+1}^{q+s}\left\{\frac{c_j^+(\tau',\eta')\mathcal{P}(\xi_j^+)}{\xi'-\xi_j^+}+R^+(\xi'\,;\tau',\eta')\right\}\tilde{g}(c\xi')d\xi'$$
$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=q+1}^{q+s'}\left\{\frac{c_j^-(\tau',\eta')\mathcal{P}(\xi_j^-)}{\xi'-\xi_j^-}+R^-(\xi'\,;\tau',\eta')\right\}\tilde{f}(c\xi')d\xi'.$$

With the help of above lemmas we can construct a compensating function $\tilde{g}(c\xi')$ from the condition

$$\sum_{j=1}^{q}\int_{-\infty}^{\infty}\left\{\frac{c_j^+(\tau',\eta')\mathcal{P}(\xi_j^+)}{\xi'-\xi_j^+}\tilde{g}(c\xi')+\frac{c_j^-(\tau',\eta')\mathcal{P}(\xi_j^+)}{\xi'-\xi_j^-}\tilde{f}(c\xi')\right\}d\xi'=0$$

and further $\tilde{g}(c\xi')$ satisfies the following properties:

1) $\int_{-\infty}^{\infty}|\tilde{g}(c\xi')|^2d\xi'\leq\text{const.}\int_{-\infty}^{\infty}|\tilde{f}(c\xi')|^2d\xi'$.

2) the support of $\tilde{g}(\tau\,;x,\eta)$ is contained in R_-^1.

Proposition 2. *Under Conditions* I *and* II, *the inequality*

$$|P\hat{v}(\tau\,;0,\eta)|\leq\frac{\text{const.}}{\sqrt{\operatorname{Re}\tau}}\left(\int_{-\infty}^{\infty}|\tilde{f}(\xi)|^2d\xi\right)^{\frac{1}{2}}$$

holds for $(\tau,\eta)\in V\cap\{\operatorname{Re}\tau>0\}$. *Where the constant does not depend on* τ *and* η.

Next we treat the solution $\hat{w}(\tau\,;x,\eta)$ in $L^2(R_+^1)$ of the problem (1.4). As $\hat{w}(\tau\,;0,\eta)$ should be in $E^+(\tau,\eta)$, $\hat{w}(\tau\,;0,\eta)$ can be written in the form

(3.15) $\hat{w}(\tau\,;0,\eta)=c_1h_1^+(\tau,\eta)+\cdots+c_mh_m^+(\tau,\eta)$

(3.16) $P\hat{w}(\tau\,;0,\eta)=c_1Ph_1^+(\tau,\eta)+\cdots+c_mPh_m^+(\tau,\eta)=-P\hat{v}(\tau\,;0.\eta)$

From Condition III and the Cramer formula

(3.17) $|c_i(\tau,\eta)|\leq\text{const.}\ |P\hat{v}(\tau\,;0,\eta)|$.

The solution $\hat{w}(\tau\,;x,\eta)$ in $L^2(R_+^1)$ of the problem (1.4) is

(4.18) $$\hat{w}(\tau\,;x,\eta)=\frac{1}{2\pi}\oint_c e^{i\xi x}(i\xi I+M(\tau,\eta))^{-1}\hat{w}(\tau\,;0,\eta)d\xi$$

where c is a simple closed curve containing the roots with positive imaginary part of $\det(\tau I+i\xi A+i\eta B)=0$ in ξ (see M. Mizohata [4]). By Proposition 2, we have

(4.19) $$\int_0^{\infty}|\hat{w}(\tau\,;x,\eta)|^2\,dx\leq\frac{\text{const.}}{(\operatorname{Re}\tau)^2}\int_{-\infty}^{\infty}|\tilde{f}(\xi)|^2d\xi.$$

This inequality and Proposition 1 follow the theorem.

The detailed proof of the theorem will appear in Journal of Mathematics of Kyoto University.

Acknowledgement. The author is deeply indebted Professor S. Mizohata [5] who suggested the problem and contributed much invaluable encouragement and advice. He wishes also to thank Professors M. Yamaguti and M. Matsumura for helpful conversations.

References

[1] Agmon, S.: Colloques internationaux du centre national de la recherche scientifique, No. 117, pp. 13–18, Paris (1962).
[2] Hersch, R.: J. Math. and Mech., **12**, 317–334 (1963).
[3] Matsumura, M.: Publ. RIMS, Kyoto Univ., Ser. A, **4**, 309–359 (1968).
[4] Mizohata, S.: Introductions to Integral Equations (Book). Asakura Pub. (1968).
[5] ——: On Agmon's results for hyperbolic equations (unpublished).
[6] Sarason, L.: Arch. Rational Mech. Anal., **18**, 311–334 (1965).