151. On Wiener Compactification of a Riemann Surface Associated with the Equation $\Delta u=pu$

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1. We consider an elliptic partial differential equation

$$(*)\qquad \Delta u=pu$$

on a Riemann surface R, where $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2$ and p is a non-negative and continuously differentiable function of local parameters z such that the expression $p(z)|dz|^2$ is invariant under the change of local parameters z. We call such a function p a density on R.

The investigation of the global theory of $(*)$ was begun by M. Ozawa [8] and continued by many others (for example, L. Myrberg [4], H. L. Royden [9], M. Nakai [5] [6] and F. Maeda [3]).

Associated with the equation $(*)$, Wiener functions and the Wiener compactification $R^*_{W^p}$ of R are discussed; more generally the Wiener compactification of harmonic spaces is studied by C. Constantinescu and A. Cornea [2]. In this note we shall examine how the Wiener compactification depends on a density p, and we shall give the following result (Theorem 4); If p and q are two densities on R satisfying

$$\text{(I)}\qquad \alpha^{-1}q\leq p\leq \alpha q$$

on R for some constant $\alpha\geq 1$, or

$$\text{(II)}\qquad \iint_R |p(z)-q(z)|\ dxdy<\infty$$

then there exists a homeomorphism Φ^* of $R^*_{W^p}$ onto $R^*_{W^q}$ such that $\Phi^*(\Gamma_{W^p})=\Gamma_{W^q}$, where Γ_{W^p} (or Γ_{W^q}) is a harmonic boundary of $R^*_{W^q}$ (or $R^*_{W^q}$).

2. Let Ω be an open subset of a Riemann surface R. A function u on Ω is called p-harmonic on Ω if u is twice continuously differentiable and satisfies $(*)$. A p-superharmonic function is defined as usual (see [3]). We know that a twice continuously differentiable function s on Ω is p-superharmonic on Ω if and only if $\Delta s-ps\leq 0$ on Ω. Let a be an arbitrary point on R. L. Myrberg [4] proved that if $p\not\equiv 0$, there exists always the Green function of R with pole at a for the equation $(*)$. We denote it by $g_a^{p,R}$.

3. A real-valued function f on R is called a p-Wiener function when f is quasicontinuous and has a p-superharmonic majorant and for any subdomain Ω, f is p-harmonizable on Ω[1]: the totality of p-Wiener functions on R is denoted by $W^p(R)$. A p-Weiner function f with $h_f^{p,R}=0$[1] is called a p-Wiener potential on R: the totality of p-Wiener potentials on R is denoted by $W_0^p(R)$. We have the following facts similarly to [1].

(a) The class $W^p(R)$ (or $W_0^p(R)$) is a vector lattice with respect to maximum and minimum.

(b) A non-negative p-superharmonic function is a p-Wiener function.

(c) A bounded p-Wiener function f has a unique decomposition $f=h_f^{p,R}+f_0$, where f_0 is a bounded p-Wiener potential on R.

(d) Let $\{R_n\}$ be an exhaustion[2] of R and f be continuous and p-harmonizable on R. Then $h_f^{p,R}=\lim_{n\to\infty} H_f^{p,R_n}$[3] on R.

(e) If f is a bounded continuous function and has the property $(V)^p$ (or $(V)_0^p$),[4] then f is a p-Wiener function (or p-Wiener potential).

4. From now on, we denote by $BW^p(R)$ the totality of bounded continuous p-Wiener functions on R and by $BH^p(R)$ the totality of bounded p-harmonic functions on R. As to the dependence of the class $BW^p(R)$ (or $BW_0^p(R)$) on p we have the following lemmas.

Lemma 1. *Let p and q be two densities on R such that $q\leq p$ on R. Then $BW_0^q(R)\subset BW_0^p(R)$ and $BW^q(R)\subset BW^p(R)$.*

Proof. Let f be a real-valued bounded function on R. Then it is easily seen that $\bar{W}^{q,R}_{\max(f,0)}\subset \bar{W}^{p,R}_{\max(f,0)}$ and $\bar{W}^{p,R}_{\max(f,0)}\subset \bar{W}_f^{p,R}$ and so that $\bar{h}^{p,R}_{\max(f,0)}\leq \bar{h}^{q,R}_{\max(f,0)}$ and $\bar{h}_f^{p,R}\leq \bar{h}^{p,P}_{\max(f,0)}$. Hence we have $\bar{h}_f^{p,R}\leq \bar{h}_f^{q,R}\vee 0$, and replacing f by $-f$, we obtain that $h_f^{p,R}\geq h_f^{q,R}\wedge 0$. By these facts we have $BW_0^q(R)\subset BW_0^p(R)$. Let f be a function in $BW^q(R)$. Then by (a), $f^+=\max(f,\ 0)$ is also a function in $BW^q(R)$. Hence by the above assertion we see that $f^+-h_{f^+}^{q,R}$ is a function in $BW_0^p(R)$. On the other hand, $h_{f^+}^{q,R}$ is a non-negative p-superharmonic function and so by (b) it is a function in $BW^p(R)$. Hence f^+ is a function in $BW^p(R)$. Similarly $f^-=\max(-f,\ 0)$ is a function in $BW^p(R)$, so that $BW^q(R)\subset BW^p(R)$.

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1) The p-harmonizability is defined analogously to the usual one: We set $\overline{w}_f^{P\Omega}=\{s;\ p\text{-superharmonic on } \Omega \text{ and } s\geq f \text{ on } \Omega\text{-}K \text{ for some compact set } K\}$, $\underline{w}_f^{P\Omega}=\{s;\ -s\in \overline{w}_{-f}^{P\Omega}\}$ and $\bar{h}_f^{P\Omega}(a)=\inf\ \{s(a);\ s\in \overline{w}_f^{P\Omega}\}$, $\underline{h}_f^{P\Omega}(a)=\sup\ \{s(a);\ s\in \underline{w}_f^{P\Omega}\}$. When $\bar{h}_f^{P\Omega}=\underline{h}_f^{P\Omega}=h_f^{P\Omega}$, we say that f is p-harmonizable on Ω; we note that $h_f^{P\Omega}$ is p-harmonic on Ω.

2) We always consider a regular exhaustion.

3) We denote by $H_f^{PR_n}$ a function continuous on $\bar{R}_n$ and p-harmonic on R_n and equal to f on ∂R_n.

4) It means that the sequence $\{H_f^{PR_n}\}$ converges (or converges to 0) for any exhaustion $\{R_n\}$ of R.

Lemma 2. *Let p be a density on R. Then $BW_0^p(R) = BW_0^{\alpha p}(R)$ for any positive constant α.*

Proof. We may assume that $0 < \alpha \leq 1$. By Lemma 1, we have only to show that $BW_0^p(R) \subset BW_0^{\alpha p}(R)$. Let f be a function in $BW_0^p(R)$. Without loss of generality we may assume that $0 \leq f \leq 1$. It is easy to see that

$$H_f^{p,R_n} \leq H_f^{\alpha p,R_n} \leq (H_f^{p,R_n})^\alpha$$

on R_n for any exhaustion $\{R_n\}$ of R. By (d), $\lim_{n\to\infty} H_f^{p,R_n} = 0$ and so f has the property $(V)_0^{\alpha p}$. Hence by (e), f is a function in $BW_0^{\alpha p}(R)$.

As to the dependence of the class $BH^p(R)$ on a density p, H. L. Royden [9] proved the following lemma.

Lemma 3. *If p and q are two densities on R satisfying the condition* (I), *then there exists an isomorphism π of $BH^p(R)$ onto $BH^q(R)$.*

We shall extend this fact to the class $BW^p(R)$.

Theorem 1. *If p and q are two densities on R satisfying the condition* (I), *then $BW^p(R)$ and $BW^q(R)$ are isomorphic.*

Proof. By Lemma 3 there exists an isomorphism π of $BH^p(R)$ onto $BH^q(R)$. Since $\alpha^{-1}q \leq p \leq \alpha q$ on R, we have $BW_0^p(R) = BW_0^q(R)$ by Lemmas 1 and 2. Let f be a function in $BW^p(R)$, then there exists uniquely a function f_0 in $BW_0^p(R)$ such that $f = h_f^{p,R} + f_0$. We define a mapping ρ as follows:

$$\rho(f) = \pi(h_f^{p,R}) + f_0$$

Then it is easy to see that ρ is an isomorphism of $BW^p(R)$ onto $BW^q(R)$.

5. M. Nakai [6] proved that if two densities p and q satisfy the condition (II), then $BH^p(R)$ and $BH^q(R)$ are isomorphic.

Using his method we shall prove the following

Theorem 2. *If p and q satisfy the condition* (II),

$$BW^p(R) = BW^q(R).$$

Proof. Let $\{R_n\}$ be an exhaustion of R. Given a real-valued bounded continuous function f on R, we define a transformation Tf as follows:

$$Tf(z_0) = f(z_0) + \frac{1}{2\pi}\iint_R (p(z) - q(z)) g_{z_0}^{q,R}(z) f(z)\, dxdy$$

We also define a transformation $T_n f$ for a function f on R_n as follow:

$$T_n f(z_0) = f(z_0) + \frac{1}{2\pi}\iint_{R_n} (p(z) - q(z)) g_{z_0}^{q,R_n}(z) f(z)\, dxdy$$

These are well-defined in virtue of the condition (II). By the Green formula we have easily that $T_n H_f^{p,R_n} = H_f^{q,R_n}$. M. Nakai [6] proved that if a uniformly bounded sequence $\{f_n\}$ of continuous functions f_n on R_n converges to a function f uniformly on each compact subset, then for each point z_0 in R

$$(**) \qquad Tf(z_0)=\lim_{n\to\infty} T_n f_n(z_0).$$

If f is a function in $BW^p(R)$, then the sequence $\{H_f^{p,R_n}\}$ is uniformly bounded and by (*d*), $\{H_f^{p,R_n}\}$ converges to $h_f^{p,R}$ uniformly on each compact subset, hence by the above assertion, $\lim_{n\to\infty} T_n H_f^{p,R_n}=Th_f^{p,R}$, so that the sequence $\{H_f^{q,R_n}\}$ converges to $Th_f^{p,R}$, namely f has the property $(V)^q$. Thus $BW^p(R)\subset BW^q(R)$. By replacing p and q we have $BW^q(R)\subset BW^p(R)$ and $BW^p(R)=BW^q(R)$.

Remark. As M. Nakai [6] remarked, (**) can be proved under the following weaker condition:

$$(\mathrm{II})' \qquad \iint_R |p(z)-q(z)|\,(g_{z_0}^{p,R}(z)+g_{z_1}^{q,R}(z))dxdy<\infty$$

for some points z_0 and z_1 in R. Therefore we have the equality $BW^p(R)=BW^q(R)$ for any p and q satisfying the condition (II)$'$.

6. Let $R_{W^p}^*$ be a $BW^p(R)$-compactification of R and Γ_{W^p} be a harmonic boundary of $R_{W^p}^*$ (cf. [1]).

As to the dependence of $R_{W^p}^*$ on a density p, we have the following fact as a corollary of Nakai's theorem (see [7]).

Theorem 3[5]**.** *Consider arbitrary two Riemann surfaces R and R'. Let p be a density on R and p' be a density on R'. If $BW^p(R)$ and $BW^{p'}(R')$ are isomorphic, then there exists a homeomorphism Φ^* of $R_{W^p}^*$ onto $R'^*_{W^{p'}}$ such that $\Phi^*(\Gamma_{W^p})=\Gamma'_{W^{p'}}$.*

By Theorems 1, 2 and 3, we have

Theorem 4. *If p and q are two densities on R satisfying the condition* (I) *or* (II), *then there exists a homeomorphism Φ^* of $R_{W^p}^*$ onto $R_{W^q}^*$ such that $\Phi^*(\Gamma_{W^p})=\Gamma_{W^q}$.*

Remark. (i) By the remark on Theorem 2 we see that the condition (II) can be replaced by the weaker condition (II)$'$.

(ii) When p or q is identically zero in the condition (II) (or (II)$'$), R is assumed to be a hyperbolic Riemann surface.

References

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5) In case that $p=p'\equiv 0$, Theorem 3 is reduced to Nakai's theorem.

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