150. *5-dimensional Orientable Submanifolds of* R^7. II

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1. Introduction. In our previous paper [4], we showed that, using the vector cross product induced by Cayley numbers, any 5-dimensional orientable submanifold M of R^7 admits an almost contact structure.

In this paper, denoting this almost contact structure by (ϕ, ξ, η), we shall study the torsion of ϕ. First, we shall prove that if M is totally geodesic then the torsion of ϕ vanishes identically (Theorem 1). Secondly, we consider the converse problem. Unfortunately, this is not true in general. But we shall prove that if M is totally umbilical, then the vanishing of the torsion of ϕ implies that M is totally geodesic (Theorem 2).

2. Basic informations.

(a) Almost contact manifolds.

Let M be a $(2n+1)$-dimensional C^∞ manifold with an almost contact structure (ϕ, ξ, η). Then we have, by definition,

$$\eta(\xi)=1, \tag{1}$$

$$\phi(\xi)=0, \qquad \eta\circ\phi=0, \tag{2}$$

$$\phi^2=-I+\eta(\cdot)\xi, \tag{3}$$

where I is the identity transformation field.

By above relations, it can be easily shown that the rank of ϕ is $2n$.

We denote the associated Riemannian metric of (ϕ, ξ, η) by $\langle\ ,\ \rangle$. Then it satisfies

$$\eta=\langle \xi, \cdot\rangle, \tag{4}$$

(5) $\langle \phi X, \phi Y\rangle=\langle X, Y\rangle-\eta(X)\eta(Y)$, for any vector fields X, Y on M.

The tensor $N(X, Y)$ defined by

$$N(X, Y)=[X, Y]+\phi[\phi X, Y]+\phi[X, \phi Y]-[\phi X, \phi Y] -\{X\cdot\eta(Y)-Y\cdot\eta(X)\}\xi \tag{6}$$

is called the *torsion* of ϕ and M is called *normal* if N vanishes identically.

(b) The vector cross product on R^7.

The vector cross product on R^7 is a linear map $P: V(R^7)\times V(R^7) \to V(R_7)$ (writing here $P(\bar{X}, \bar{Y})=\bar{X}\otimes\bar{Y}$) satisfing the following conditions:

$$\bar{X}\otimes\bar{Y}=-\bar{Y}\otimes\bar{X}, \tag{7}$$

$$\langle \bar{X}\otimes\bar{Y}, \bar{Z}\rangle=\langle \bar{X}, \bar{Y}\otimes\bar{Z}\rangle, \tag{8}$$

(9) $(\bar{X}\otimes\bar{Y})\otimes\bar{Z}+X\otimes(\bar{Y}\otimes\bar{Z})=2\langle\bar{X},\bar{Z}\rangle\bar{Y}-\langle\bar{Y},\bar{Z}\rangle\bar{X}-\langle\bar{X},\bar{Y}\rangle\bar{Z},$

(10) $\bar{\nabla}_{\bar{X}}(\bar{Y}\otimes\bar{Z})=\bar{\nabla}_{\bar{X}}\bar{Y}\otimes\bar{Z}+\bar{Y}\otimes\bar{\nabla}_{\bar{X}}\bar{Z},$

where $V(R^7)$ is the ring of differentiable vector fields on R^7, $\bar{X},\bar{Y},\bar{Z}\in V(R^7)$ and $\bar{\nabla}$ is the covariant differentiation of R^7.

3. 5-dimensional orientable totally geodesic and totally umbilical submanifolds of R^7.

Let M be a 5-dimensional orientable submanifold of R^7. Then there exist locally defined mutually orthogonal differentiable unit normal vector fields C_1, C_2 to M.

For any X, $Y\in V(M)$, we can put

(11) $$\begin{cases}\bar{\nabla}_X C_1=-A_1X+s(X)C_2\\ \bar{\nabla}_X C_2=-A_2X-s(X)C_1,\end{cases}$$

where $-A_1X$ (resp. $-A_2X$) is the tangential part of $\bar{\nabla}_XC_1$ (resp. $\bar{\nabla}_XC_2$) and s is a 1-form on M.

Then the *equation of Weingarten* can be taken of the form

(12) $$\bar{\nabla}_XY=\nabla_XY+\langle A_1X,Y\rangle C_1+\langle A_2X,Y\rangle C_2,$$

where ∇_XY is the tangential part of $\bar{\nabla}_XY$. It is well known that ∇ is the covariant differentiation of M with respect to the induced Riemannian metric and A_1, A_2 are symmetric (1,1) type tensors (e.g. [3]).

We put

(13) $$\xi=C_1\otimes C_2,$$

(14) $$\eta(X)=\langle C_1\otimes C_2, X\rangle,$$

(15) $$\phi(X)=X\otimes(C_1\otimes C_2).$$

Then, as we showed in [4], $(\phi,\xi,\eta,\langle\ ,\ \rangle)$ gives an almost contact metric structure on M.

Proposition 1. *For $\xi=C_1\otimes C_2$ and any X, $Y\in V(M)$, we have the following identities*:

(16) $$(\bar{\nabla}_XY\otimes\xi)\otimes\xi=\langle\bar{\nabla}_XY,\xi\rangle\xi-\bar{\nabla}_XY.$$

(17) $$N(X,Y)=(Y\otimes\bar{\nabla}_X\xi-X\otimes\bar{\nabla}_Y\xi)\otimes\xi+X\otimes\bar{\nabla}_{\phi Y}\xi-Y\otimes\bar{\nabla}_{\phi X}\xi$$
$$+(\langle X,\nabla_Y\xi\rangle-\langle Y,\nabla_X\xi\rangle)\xi.$$

Proof. For (16), we have

$$\begin{aligned}(\bar{\nabla}_XY\otimes\xi)\otimes\xi&=(\bar{\nabla}_XY\otimes\xi)\otimes\xi+\bar{\nabla}_XY\otimes(\xi\otimes\xi)\\&=2\langle\bar{\nabla}_XY,\xi\rangle\xi-\langle\xi,\xi\rangle\bar{\nabla}_XY-\langle\bar{\nabla}_XY,\xi\rangle\xi \qquad \text{(by (9))}\\&=\langle\bar{\nabla}_XY,\xi\rangle\xi-\bar{\nabla}_XY.\end{aligned}$$

For (17), we have

$$\begin{aligned}N(X,Y)=&[X,Y]+\phi[\phi X,Y]+\phi[X,\phi Y]-[\phi X,\phi Y]\\&-\{X\cdot\eta(Y)-Y\cdot\eta(X)\}\xi\\=&\bar{\nabla}_XY-\bar{\nabla}_YX+\bar{\nabla}_{\phi X}Y\otimes\xi-\bar{\nabla}_Y\phi X\otimes\xi+\bar{\nabla}_X\phi Y\otimes\xi\\&-\bar{\nabla}_{\phi Y}X\otimes\xi-\bar{\nabla}_{\phi X}\phi Y+\bar{\nabla}_{\phi Y}\phi X-\{X\cdot\eta(Y)-Y\cdot\eta(X)\}\xi\\=&\bar{\nabla}_XY-\bar{\nabla}_YX+\bar{\nabla}_{\phi X}Y\otimes\xi-(\bar{\nabla}_YX\otimes\xi)\otimes\xi-(X\otimes\bar{\nabla}_Y\xi)\otimes\xi\\&+(\bar{\nabla}_XY\otimes\xi)\otimes\xi+(Y\otimes\bar{\nabla}_X\xi)\otimes\xi-\bar{\nabla}_{\phi Y}X\otimes\xi-\bar{\nabla}_{\phi X}Y\otimes\xi\\&-Y\otimes\bar{\nabla}_{\phi X}\xi+\bar{\nabla}_{\phi Y}X\otimes\xi+X\otimes\bar{\nabla}_{\phi Y}\xi-\{X\cdot\eta(Y)-Y\cdot\eta(X)\}\xi\end{aligned}$$

$$
\begin{aligned}
&=\bar{\nabla}_X Y-\bar{\nabla}_Y X-(\langle\bar{\nabla}_Y X,\, \xi\rangle\xi-\bar{\nabla}_Y X)-(X\otimes\bar{\nabla}_Y\xi)\otimes\xi\\
&\quad+(\langle\bar{\nabla}_X Y,\, \xi\rangle\xi-\bar{\nabla}_X Y)+(Y\otimes\bar{\nabla}_X\xi)\otimes\xi-Y\otimes\bar{\nabla}_{\phi X}\xi\\
&\quad+X\otimes\bar{\nabla}_{\phi Y}\xi-(\langle\nabla_X Y,\, \xi\rangle+\langle Y,\, \nabla_X\xi\rangle-\langle\nabla_Y X,\, \xi\rangle\\
&\quad-\langle X,\, \nabla_Y\xi\rangle)\xi \qquad \text{(by (16))}\\
&=(Y\otimes\bar{\nabla}_X\xi-X\otimes\bar{\nabla}_Y\xi)\otimes\xi+X\otimes\bar{\nabla}_{\phi Y}\xi-Y\otimes\bar{\nabla}_{\phi X}\xi\\
&\quad+(\langle X,\, \nabla_Y\xi\rangle-\langle Y,\, \nabla_X\xi\rangle)\xi.
\end{aligned}
$$

Q.E.D.

Proposition 2. *For $\xi=C_1\otimes C_2$ and any $X\in V(M)$, we have*

$$\bar{\nabla}_X\xi=-A_1X\otimes C_2+A_2X\otimes C_1, \tag{18}$$

so that consequently

$$\nabla_X\xi+\langle A_1X,\, \xi\rangle C_1+\langle A_2X,\, \xi]C_2=-A_1X\otimes C_2+A_2X\otimes C_1 \tag{19}$$

holds good.

Proof. For (18), we have by (10) and (11),

$$
\begin{aligned}
\bar{\nabla}_X\xi&=\bar{\nabla}_X(C_1\otimes C_2)\\
&=\bar{\nabla}_X C_1\otimes C_2+C_1\otimes\bar{\nabla}_X C_2\\
&=(-A_1X+s(X)C_2)\otimes C_2+C_1\otimes(-A_2X-s(X)C_1)\\
&=-A_1X\otimes C_2+s(X)C_2\otimes C_2-C_1\otimes A_2X-C_1\otimes s(X)C_1\\
&=-A_1X\otimes C_2+A_2X\otimes C_1.
\end{aligned}
$$

And, replacing Y by ξ in (12) we have the left hand side of (19), from which (19) follows immediately. Q.E.D.

Theorem 1. *Let M be a 5-dimensional orientable totally geodesic submanifold of R^7. Then the torsion of ϕ vanishes identically.*

Proof. Since M is totally geodesic, we have $A_1=A_2=0$, which implies $\bar{\nabla}_X\xi=0$ by (18) of Proposition 2. Hence we have $N=0$ by (17) of Proposition 1. Q.E.D.

Proposition 3. *For $\xi=C_1\otimes C_2$, we have the following identities:*

$$C_1\otimes\xi=-C_2. \tag{20}$$

$$C_2\otimes\xi=C_1. \tag{21}$$

Proof. For (20), we have

$$
\begin{aligned}
C_1\otimes\xi&=C_1\otimes(C_1\otimes C_2)\\
&=C_1\otimes(C_1\otimes C_2)+(C_1\otimes C_1)\otimes C_2\\
&=2\langle C_1,\, C_2\rangle C_1-\langle C_1,\, C_2\rangle C_1-\langle C_1,\, C_1\rangle C_2\\
&=-C_2.
\end{aligned}
$$

Similarly, we have $C_2\otimes\xi=C_1$. Q.E.D.

Theorem 2. *Let M be a 5-dimensional orientable totally umbilical submanifold of R^7. If the torsion of ϕ vanishes identically, then M is totally geodesic.*

Proof. Making an inner product $N(X, Y)$ with ξ, and using (8), we have

$$\langle X,\, \bar{\nabla}_{\phi Y}\xi\otimes\xi\rangle-\langle Y,\, \bar{\nabla}_{\phi X}\xi\otimes\xi\rangle+\langle X,\, \nabla_Y\xi\rangle-\langle Y,\, \nabla_X\xi\rangle=0. \tag{22}$$

On the other hand, since M is totally umbilical we have $A_1=\lambda_1 I$ and $A_2=\lambda_2 I$, for some scalars λ_1 and λ_2. Hence, we have by (18),

$$\bar{\nabla}_{\phi Y}\xi=-\lambda_1\phi Y\otimes C_2+\lambda_2\phi Y\otimes C_1.$$

Thus, we have

$$
\begin{aligned}
\bar{\nabla}_{\phi Y}\xi\otimes\xi &= (-\lambda_1\phi Y\otimes C_2+\lambda_2\phi Y\otimes C_1)\otimes\xi \\
&= -\lambda_1(\phi Y\otimes C_2)\otimes\xi+\lambda_2(\phi Y\otimes C_1)\otimes\xi \\
&= -\lambda_1\{2\langle\phi Y,\xi\rangle C_2-\langle C_2,\xi\rangle\phi Y-\langle\phi Y, C_2\rangle\xi-\phi Y\otimes(C_2\otimes\xi)\} \\
&\quad +\lambda_2\{2\langle\phi Y,\xi\rangle C_1-\langle C_1,\xi\rangle\phi Y-\langle\phi Y, C_1\rangle\xi-\phi Y\otimes(C_1\otimes\xi)\} \\
&\qquad\qquad \text{(by (9))} \\
&= \lambda_1\phi Y\otimes(C_2\otimes\xi)-\lambda_2\phi Y\otimes(C_1\otimes\xi) \\
&= \lambda_1(Y\otimes\xi)\otimes C_1+\lambda_2(Y\otimes\xi)\otimes C_2 \qquad \text{(by (15), (20) and (21))} \\
&= \lambda_1\{-\langle Y,\xi\rangle C_1-Y\otimes(\xi\otimes C_1)\}+\lambda_2\{-\langle Y,\xi\rangle C_2-Y\otimes(\xi\otimes C_2)\} \\
&\qquad\qquad \text{(by (9))} \\
&= -\lambda_1\langle Y,\xi\rangle C_1-\lambda_1 Y\otimes C_2-\lambda_2\langle Y,\xi\rangle C_2+\lambda_2 Y\otimes C_1. \\
&\qquad\qquad \text{(by (20) and (21))}
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\langle X, \bar{\nabla}_{\phi Y}\xi\otimes\xi\rangle &= \langle X, -\lambda_1\langle Y,\xi\rangle C_1-\lambda_1 Y\otimes C_2-\lambda_2\langle Y,\xi\rangle C_2+\lambda_2 Y\otimes C_1\rangle \\
&= \langle X, -\lambda_1 Y\otimes C_2+\lambda_2 Y\otimes C_1\rangle \\
&= \langle X, \bar{\nabla}_Y\xi\rangle \\
&= \langle X, \nabla_Y\xi\rangle.
\end{aligned}
$$

Similarly, we have $\langle Y, \bar{\nabla}_{\phi X}\xi\otimes\xi\rangle=\langle Y, \nabla_X\xi\rangle$.

Therefore, (22) reduces to

(23) $$\langle X, \nabla_Y\xi\rangle-\langle Y, \nabla_X\xi\rangle=0.$$

But, on the other hand, we have

$$
\begin{aligned}
\langle X, \nabla_Y\xi\rangle+\langle Y, \nabla_X\xi\rangle &= \langle X, -\lambda_1 Y\otimes C_2+\lambda_2 Y\otimes C_1\rangle \\
&\quad +\langle Y, -\lambda_1 X\otimes C_2+\lambda_2 X\otimes C_1\rangle \\
&= -\lambda_1\langle X\otimes Y+Y\otimes X, C_2\rangle+\lambda_2\langle X\otimes Y+Y\otimes X, C_1\rangle \\
&= 0,
\end{aligned}
$$

which, together with (23), implies $\nabla_X\xi=0$.

Thus, from (19), we have

$$\lambda_1\langle X,\xi\rangle C_1+\lambda_2\langle X,\xi\rangle C_2=-\lambda_1 X\otimes C_2+\lambda_2 X\otimes C_1.$$

Applying $\otimes C_1$ from the right on both sides of this equation, we have

$$
\begin{aligned}
\lambda_2\langle X,\xi\rangle C_2\otimes C_1 &= -\lambda_1(X\otimes C_2)\otimes C_1+\lambda_2(X\otimes C_1)\otimes C_1 \\
&= \lambda_2 X\otimes(C_2\otimes C_1)-\lambda_2 X \qquad \text{(by (9))}
\end{aligned}
$$

that is,

(24) $$\lambda_2\langle X,\xi\rangle\xi=\lambda_1 X\otimes\xi+\lambda_2 X.$$

Making an inner product (24) with X, we have

$$\lambda_2\langle X,\xi\rangle^2=\lambda_2\langle X, X\rangle.$$

Since $\eta(X)=\langle X,\xi\rangle$, the above equation reduces to

$$\lambda_2\langle\phi X, \phi X\rangle=0,$$

by virtue of (5).

Since the rank of ϕ is 4 and $\langle\ ,\ \rangle$ is a Riemannian metric, we can conclude $\lambda_2=0$.

Similarly, we have $\lambda_1=0$, which shows that M is totally geodesic.

Q.E.D.

References

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