144. *A Class of Markov Processes with Interactions. I*[1)]

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Here, we consider a motion of one particle under the interactions between an infinite number of similar particles. Each particle moves independently in a Markovian way until an exponential jumping time comes, and it jumps with a hitting measure which depends on other particles. A model, where the jumping time also depends on other particles, is discussed under auxiliary conditions. These results extend [9].

The models here came into our interest through the works of McKean [3–5], which started with Kac's model of Boltzmann equation [2].

1. Let $P(s, x, t, E)$ be a transition probability on a locally compact space R with countable bases and topological Borel field $\boldsymbol{B}(R)$. Assume $P(s, x, t, R) \equiv 1$ and

(1) $\qquad P(s, x, t, U) \to 1, \quad$ as $t-s \to 0$, for open U containing x.

Let $q(t, y)$ be a non-negative, measurable function, bounded on compact (t, y)-sets. Define

(2) $$P_0(s, x, t, E) = E_{s,x}\left(\exp\left[-\int_s^t q(\sigma, X_\sigma(w))d\sigma\right]\chi_E(X_t(w))\right),$$

where $X_t(w)$ is a measurable Markov process with transition probability $P(s, x, t, E)$. $E_{s,x}(\cdot)$ is the expectation conditioned that the particle starts at x at time s. This set up is possible by (1). Let $q_n(t, y)$, $n=0, 1, \cdots$ be non-negative, measurable and $q(t, y) \equiv \sum_{n=0}^\infty q_n(t, y)$, and let $\pi_n(y_1, \cdots, y_n \mid t, y)$ be probability measures on $(R, \boldsymbol{B}(R))$, measurable in $(y_1, \cdots, y_n, t, y)$ for fixed $E \in \boldsymbol{B}(R)$.[2)]

Consider a *forward equation* and a version of *backward equation*:

(3) $$\begin{aligned} P^{(f)}&(s, x, t, E) \\ &= P_0(s, x, t, E) + \int_s^t d\tau \int_R P^{(f)}(s, x, \tau, dy) \sum_{n=0}^\infty q_n(t, y) \int_{R^n} \prod_{k=1}^n P_{s,\tau}^{(f)}(dy_k) \\ &\quad \times \int_R \pi_n(y_1, \cdots, y_n \mid \tau, y, dz)\, P_0(\tau, z, t, E),^{3)} \end{aligned}$$

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2) For the intuitive meanings of the quantities, the reader can consult [9].

3) The 0-th term of the sum is $q_0(\tau, y) \int_R \pi_0(\tau, y, dz)\, P_0(\tau, z, t, E)$.

$$(4)\quad P^{(P^{(f)}_{s0s})}(s,x,t,E)$$
$$=P_0(s,x,t,E)+\int_s^t d\tau\int_R P_0(s,x,\tau,dy)\sum_{n=0}^{\infty} q_n(\tau,y)$$
$$\times\int_{R^n}\prod_{k=1}^{n} P^{(f)}_{s0\tau}(dy_k)\int_R \pi_n(y_1,\cdots,y_n\mid\tau,y,dz)P^{(P^{(f)}_{s0\tau})}(\tau,z,t,E),$$

where f is a substochastic measure[4] on R and

$$P^{(f)}_{s,\tau}(E)=\int_R f(dx)P^{(f)}(s,x,\tau,E).$$

Theorem 1. i) *Forward equation* (3) *has the minimal substochastic solution* $P^{(f)}(s,x,t,E)$. ii) $P^{(f)}(s,x,t,E)$ *satisfies a version of Chapman-Kolmogorov equation*:

$$(5)\quad P^{(f)}(s,x,u,E)=\int_R P^{(f)}(s,x,t,dy)P^{(P^{(f)}_{s,t})}(t,y,u,E),\qquad s\le t\le u.$$

iii) $P^{(f)}(s,x,t,E)$ *satisfies* (4), *and is also the minimal among substochastic solutions of* (4). iv) *If the minimal solution is a probability measure, it is the unique solution of* (3) *and* (4). *This occurs when the following* a) *or* b) *holds and* $f(R)=1$.

a) *There are* $q_n(t)$*'s such that* $q_n(t,y)\le q_n(t)$ *and* $\sum_{n=0}^{\infty} nq_n(t)$ *is locally* L^α, $\alpha>1$.

b) *There are constants* q_n*'s such that* $q_n(t,y)\le q_n$, $\sum_{n=0}^{\infty} q_n<\infty$, *and*

$$(6)\quad \int_{1-\varepsilon}^{1}\left(\sum_{n=1}^{\infty} q_n(\tau-\tau^{n+1})\right)^{-1}d\tau=\infty,\qquad \text{for}\quad 0<\varepsilon<1.$$

Proofs of i), ii) and a part of iii) are parallel to [9], using

$$\int_s^t d\tau\int_R P_0(s,x,\tau,dy)q(\tau,z)=1-P_0(s,x,t,R)\le 1-P_0(s,x,t,E).$$

iv) To prove $P^{(f)}(s,x,t,R)\equiv 1$ when a) holds, let $S^{(f)}_m$ be the m-th approximation to the minimal solution $P^{(f)}$, that is, $S^{(f)}_0(s,x,t,E)\equiv P_0(s,x,t,E)$ and

$$S^{(f)}_{m+1}(s,x,t,E)$$
$$=P_0(s,x,t,E)+\int_s^t d\tau\int_R S^{(f)}_m(s,x,\tau,dy)\sum_{n=0}^{\infty} q_n(\tau,y)$$
$$\times\int_{R^n}\prod_{k=1}^{n} S^{(f)}_m(s,\tau,dy_k)\int_R \pi_n(y_1,\cdots,y_n\mid\tau,y,dz)P_0(\tau,z,t,E),$$
$$S^{(f)}_m(s,\tau,E)=\int_R f(dx)S^{(f)}_m(s,x,\tau,E).$$

Then, integrate $q(t,y)$ on R by both sides of this to get

$$1-S^{(f)}_{m+1}(s,x,t,R)$$
$$=\int_s^t dz\int_R (S^{(f)}_{m+1}(s,x,\tau,dz)-S^{(f)}_m(s,x,\tau,dy))q(z,y)$$
$$+\int_s^t dz\int_R S^{(f)}_m(s,x,\tau,dy)\sum_{n=1}^{\infty} q_n(\tau,y)(1-S^{(f)}_m(s,\tau,R)^n).$$

Since $q(t)=\sum_{n=0}^{\infty} q_n(t)$ is locally L^α by a), the first term is bounded by

4) A measure is called stochastic (substochastic), if it has total mass 1 (not more than 1).

$$\int_s^t (S_{m+1}^{(t)}(s, x, \tau, R) - S_m^{(f)}(s, x, \tau, R)) q(\tau) d\tau$$
$$\leq \left(\int_s^t (S_{m+1}^{(f)} - S_m^{(f)})(s, x, \tau, R)^{\alpha/(\alpha-1)} \right)^{(\alpha-1)/\alpha} \left(\int_s^t q(\tau)^\alpha d\tau \right)^{1/\alpha} \to 0.$$

This implies

$$(7) \qquad 1 - P^{(f)}(s, x, t, R) = \int_s^t dz \int_R P^{(f)}(s, x, \tau, dy) \sum_{n=1}^\infty p_n(\tau, y)(1 - P_{s,\tau}^{(f)}(R)^n).$$

Hence, it is enough to prove $P_{s,\tau}^{(f)}(R) \equiv 1$. Integrating (7) by f and putting $g(\tau) \equiv P_{s,\tau}^{(f)}(R)$,

$$1 - g(t) = \int_s^t d\tau \int_R P_{s,\tau}^{(f)}(dy) \sum_{n=1}^\infty q_n(\tau, y)(1 - g(\tau)^n)$$
$$\leq \int_s^t d\tau \sum_{n=1}^\infty q_n(\tau)(g(\tau) - g(\tau)^{n+1}) \leq \int_s^t d\tau (1 - g(\tau)) \left(\sum_{n=1}^\infty n q_n(\tau) \right).$$

Then, it is easy to prove $1 - g(t) = 0$, since $\sum_{n=1}^\infty n q_n(\tau)$ is locally L^α, $\alpha > 1$. When $q_n(\tau, y)$'s are constants, (7), integrated by f on R, reduces to

$$(8) \qquad 1 - g(t) = \int_s^t d\tau \sum_{n=1} q_n \cdot (g(t) - g(t)^{n+1}), \qquad t \geq s.$$

But, this has 1 as a unique solution if and only if (6) is true. Hence, in case b) holds, we modify (3) to an equation with constant q_n's and $\tilde{\pi}_n$'s modified as in **3** later. Then, the minimal solution of this equation has total mass 1 and it is the minimal solution of (3). The proof of the rest of iii) is omitted here.

2. Given a forward equation of integro-differential type:

$$(9) \qquad \begin{aligned} &\frac{\partial}{\partial t} \int_R P^{(f)}(s, x, t, dy) \varphi(y) = \int_R P^{(f)}(s, x, t, dy) B_t^{(f)} \varphi(y),^{6)} \\ &\int_R P^{(f)}(s, x, t, dy) \varphi(y) \to \varphi(x), \qquad \text{as } t \downarrow s, \end{aligned}$$

$$(10) \qquad B_t^{(f)} \varphi(y) = A_t \varphi(y) + \sum_{n=0}^\infty q_n(t, y) \times \left(\int_{R^n} \prod_{k=1}^n P_{s,t}^{(f)}(dy_k) \int_R \pi_n(y_1, \cdots, y_n | t, y, dz) \varphi(z) - \varphi(y) \right)$$

where A_t is the generator[7] of $P(s, x, t, E)$ in **1**. Then, solutions of (3) solve this as in

Theorem 2. *Assume* c) $q_n(t, y)$, $q(t, y)$ *and* $\pi_n(y_1, \cdots, y_n | t, y, E)$ *are continuous in t when other variables are fixed.* $q(t, y)$ *is bounded.* d) $P(s, x, t, E)$ *is continuous in* $t(>s)$ *for fixed* $s, x, E \in \boldsymbol{B}(R)$. *For a bounded continuous function* φ, *there is a bounded function* $A_t \varphi(x)$, *continuous in x and in t, such that*

5) This condition was adopted by H. Tanaka [6] and S. Tanaka [7] for a temporally homogeneous model. The relation between (6) and (8) owes to Dynkin. The reader can consult Harris [1] p. 106 for the proof.

6) H. Tanaka wrote to the author that he considered a similar equation related with [6].

7) Here, the term *generator* is used loosely, instead of the expression in Theorem 2.

(11) $$\frac{\partial}{\partial t}\int_R P(s,x,t,dy)\varphi(y)=\int_R P(s,x,t,dy)A_t\varphi(y).$$

Then, any substochastic solution of (3) *satisfies* (9) *for this* φ.

Examples. 1) Let A_t be an elliptic operator with smooth, bounded coefficients:

$$A_t\varphi(x)=\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}\varphi(x)+\sum_{i=1}^{n}b_i(x)\frac{\partial}{\partial x_i}\varphi(x),$$
$$x=(x_1,\cdots,x_n)\in E^n.$$

Then, A_t uniquely determines $P(s,x,t,E)$ which satisfies d) for each sufficiently smooth bounded φ with bounded derivatives up to the second order.

2) Let $P(s,x,t,E)$ be temporally homogeneous,

$$T_t\varphi(x)=\int_R P(s,x,s+t,dy)\varphi(y)$$

map $B(R)$ into $C(R)$, and the semigroup $\{T_t\}$ acting on $C(R)$ be strongly continuous in t. Then d) holds for the Hille-Yosida generator $A_t\equiv A$ of $\{T_t\}$ and each φ in $D(A)$.[8]

3) When $P(s,x,t,E)=\delta_x(E)$, d) holds for each $\varphi\in B(R)$. This is the model in [9], except that g is bounded.

3. With the same initial condition of (7), consider

(9') $$\frac{\partial}{\partial t}\int_R P^{(f)}(s,x,t,dy)\varphi(y)=\int_R P^{(f)}(s,x,t,dy)C_t^{(f)}\varphi(y),$$ [9]

(10') $$C_t^{(f)}\varphi(y)=A_t\varphi(y)+\sum_{n=0}^{\infty}\int_{R^n}\prod_{k=1}^{n}P_{s,t}^{(f)}(dy_k)q_n(y_1,\cdots,y_n,t,y)\times\int_R(\pi_n(y_1,\cdots,y_n\,|\,t,y,dz)-\delta_y(dz))\varphi(z)$$

where $q_n(y_1,\cdots,y_n,t,y)$'s are non-negative and measurable. This corresponds to a model where the jumping time also depends on other particles. Here, (3) and (4) are replaced by

(3') $$P^{(f)}(s,x,t,E)=P_0^{(f)}(s,x,t,E)+\int_s^t d\tau\sum_{n=0}^{\infty}\int_{R^{n+1}}^{P(f)}P^{(f)}(s,x,\tau,dy)\times\prod_{k=1}^{n}P_{s,\tau}^{(f)}(dy_k)q_n(y_1,\cdots,y_n,\tau,y)\times\int_R\pi_n(y_1,\cdots,y_n\,|\,\tau,y,dz)P_0^{(f)}(\tau,z,t,E)$$

(4') $$P^{(P_{s_0s}^{(f)})}(s,x,t,E)=P_0^{(P_{s_0s}^{(f)})}(s,x,t,E)+\int_s^t d\tau\sum_{n=0}^{\infty}\int_{R^{n+1}}P_0^{(P_{s_0s}^{(f)})}(s,x,\tau,dy)\times\prod_{k=1}^{n}P_{s_0\tau}^{(f)}(dy_k)q_n(y_1,\cdots,y_n,\tau,y)\times\int_R\pi_n(y_1,\cdots,y_n\,|\,\tau,y,dz)P^{(P_{s_0s}^{(f)})}(\tau,z,t,E)$$

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8) $B(R)$ and $C(R)$ are the set of all real-valued functions on R, measurable and continuous, respectively. $D(A)$ is the domain of A.

9) Boltzmann equation with bounded cross section can be rewritten in this form.

where

$$P_0^{(f)}(s, x, \tau, E)=E_{s,x}\left(\exp\left[-\int_s^t q^{(f)}(s, \sigma, X_\sigma)d\sigma\right]\chi_E(X_t)\right),$$

$$q^{(f)}(s, t, y)=\sum_{n=0}^{\infty}\int_{R^n}\prod_{k=1}^{n}P_{s,t}^{(f)}(dy_k)q_n(y_1, \cdots, y_n, t, y).$$

Theorem 3. *Assume that there are measurable functions $q_n(t, y)$ such that*

$$q_n(y_1, \cdots, y_n, t, y)\leq q_n(t, y)$$

and that $q(t, y)=\sum_{n=0}^{\infty} q_n(t, y)$ is bounded on compact (t, y)-sets. i) *If $q_n(t, y)$'s satisfy* a) *or* b) *in Theorem* 1, *then* (3′) *has one and only one stochastic solution for each probability measure f. This solution solves* (4′) *and satisfies the Chapman-Kolmogorov equation* (5). ii) *Assume, moreover, the conditions for $q_n(t, y)$, $q(t, y)$, π_n and φ in Theorem* 2. *Then, this solution satisfies* (9′) *for this φ.*

It can be proved that the minimal solution of (3), with above $q_n(t, y)$ and π_n replaced by

$$\begin{aligned}\tilde{\pi}_n(y_1, \cdots, y_n | t, y, E)\\ =q(t, y)^{-1}\{q_n(y_1, \cdots, y_n, t, y)\pi_n(y_1, \cdots, y_n | t, y, E)\\ +(q_n(t, y)-q_n(y_1, \cdots, y_n, t, y))\delta_y(E)\},\end{aligned}\tag{11}$$

is the unique stochastic solution of (3′) and solves (4′). By the conditions in ii), this solves (9) with π_n replaced by $\tilde{\pi}_n$ of (11), which coincides with (9′) by $P_{s,\tau}^{(f)}(R)\equiv 1$.[10)]

4. Another extension of **1** is as follows. Let $P_0(s, x, t, E)$ be a transition probability, majorized by $P(s, x, t, E)$ satisfying (1) and $P(s, x, t, R)\equiv 1$, such that

$$0<P_0(s, x, t, R)<1, \qquad \text{for} \quad s<t.$$

Let $K_0(s, x, \Lambda)$ be a probability measure on $I\times R$ concentrated on $((s, \infty)\cap I)\times R$, where I is the interval of time parameters. Let $K_0(s, x, \Lambda)$ be measurable in (s, x) and satisfy

$$K_0(s, x, \Lambda\cap(I_t\times R))=\int_R P_0(s, x, t, dy)K_0(t, y, \Lambda), \qquad I_t=[t, \infty)\cap I.\tag{12}$$

Then, the alternative for the forward equation (3) is a pair of equations:

$$\begin{aligned}P^{(f)}(s, x, t, E)=P_0(s, x, t, E)+\int_{[s,t]\times R}K^{(f)}(s, x, d\tau, dy)\\ \times\sum_{n=0}^{\infty}p_n(\tau, y)\int_{R^n}\prod_{k=1}^{n}P_{s,\tau}^{(f)}(dy_k)\\ \times\int_R\pi_n(y_1, \cdots, y_n | \tau, y, dz)P_0(\tau, z, t, E),\end{aligned}\tag{13}$$

10) When $P_{s,t}^{(f)}(R)\neq 1$, this method does not work. The author wrote in I of [9] that an equation of type (3) seemed more natural than (3′). This should be corrected as follows: Both equations of type (3) and (3′) have nice probabilistic meanings, and a nicer method should be found for (3′) when there are no $q_n(t, y)$ as in Theorem 2, or the solution of type (3) fails to be a probability measure.

$$(14)\quad K^{(f)}(s,x,\Lambda)=K_0(s,x,\Lambda)+\int_{I_s\times R}K^{(f)}(s,x,d\tau,dz)\sum_{n=0}^{\infty}p_n(\tau,y)\times\int_{R^n}\prod_{k=1}^{n}P_{s,\tau}^{(f)}(dy_k)\int_R \pi_n(y_1,\cdots,y_n\,|\,\tau,y,dz)K_0(\tau,z,\Lambda),$$

where $p_n(t,y)$'s are non-negative and $\sum_{n=0}^{\infty}p_n(t,y)\equiv 1$. The alternative for (4) is

$$(15)\quad P^{(P_{s_0s}^{(f)})}(s,x,t,E)=P_0(s,x,t,E)+\int_{[s,t]\times R}K_0(s,x,d\tau,dy)\sum_{n=0}^{\infty}p_n(\tau,y)\int_{R^n}\prod_{k=1}^{n}P_{s_0\tau}^{(f)}(dy_k)\times\int_R \pi_n(y_1,\cdots,y_n\,|\,\tau,y,dz)P^{(P_{s_0s}^{(f)})}(\tau,y,t,E).$$

This amounts to let the particles jump according to a multiplicative functional, not necessarily of type $\exp\left(-\int_s^t q(\sigma,X_\sigma)d\sigma\right)$. In case of **1**, $p_n(\tau,y)=q_n(\tau,y)/q(\tau,y)$.

Theorem 4. i) *There is a pair of substochastic measures* $P^{(f)}(s,x,t,E)$ *and a* σ*-finite measure* $K^{(f)}(s,x,\Lambda)$ *on* $I\times R$ *concentrated on* $((s,\infty)\cap I)\times R$, *which solves* (13)–(14) *and is the minimal among all such pairs.* ii) $P^{(f)}(s,x,t,E)$ *satisfies the Chapman-Kolmogorov equation* (5) *and*

$$(16)\qquad K^{(f)}(s,x,\Lambda\cap(I_t\times R))=\int_R P^{(f)}(s,x,t,dy)K^{(P_{s,t}^{(f)})}(t,y,\Lambda).$$

iii) $P^{(f)}(s,x,t,E)$ *is also the minimal substochastic solution of* (15). iv) *If* $P^{(f)}(s,x,t,R)=1$, *then the minimal pair gives the unique solution of* (13)–(14) *and* (15). *This holds, if*

$$\int_{[s,t]\times R}K^{(f)}(s,x,d\tau,dy)\sum_{n=1}^{\infty}np_n(\tau,y)<\infty,\quad t\geq s,\quad \textit{and}\quad f(R)=1.$$

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