16. Note on the Archimedean Property in an Ordered Semigroup

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(Comm. by Kinjirô KUNUGI, M. J. A., Jan. 12, 1970)

By an *ordered semigroup* we mean a semigroup with a simple order which is compatible with the semigroup operation. In this note we denote by S an ordered semigroup. An element x of S is called *positive* if $x<x^2$, and is called *negative* if $x^2<x$. For an element x of S, the number of distinct natural powers of x is called the *oder* of x.

In [3], we studied some properties of the archimedean equivalence in an ordered semigroup in which every element is non-negative. In this note, we define the archimedean equivalence $\mathcal{A}$ in a general ordered semigroup and show that similar results hold in this general case.

Definition. *The archimedean equivalence $\mathcal{A}$ on S is defined by: for $x, y \in S$, $x\mathcal{A}y$ if and only if there exist natural numbers p, q, r and s such that $x^p \leqq y^q$ and $y^r \leqq x^s$.*

Theorem 1. *The archimedean equivalence $\mathcal{A}$ on S is an equivalence relation on S. Each $\mathcal{A}$-class is a convex subsemigroup of S.*

Lemma 2. *Each $\mathcal{A}$-class contains at most one idempotent.*

Theorem 3. *For an $\mathcal{A}$-class C, the following conditions are equivalent:*

(1) *C contains an idempotent;*

(2) *the set of all nonnegative elements of C is nonempty and has the greatest element;*

(3) *the set of all nonpositive elements of C is nonempty and has the least element;*

(4) *C has the zero element;*

(5) *every element of C is an element of finite order;*

(6) *C contains an element of finite order;*

(7) *C contains at least one nonnegative and at least one nonpositive element.*

Moreover, under these conditions, an idempotent of C is the greatest nonnegative element, the least nonpositive element and also the zero element of C.

Corollary 4. *Let x be a nonnegative element and y be an element of an $\mathcal{A}$-class C of S. Then*

(1) *$y \leqq xy$ if and only if y is nonnegative;*

(2) *$y \leqq yx$ if and only if y is nonnegative.*

Definition. *An $\mathcal{A}$-class C of S is called periodic, if one of the conditions* (1)-(7) *in Theorem* 3 *holds in C.*

Theorem 5. *Let C be a periodic $\mathcal{A}$-class of S and let e be the uniquely determined idempotent element of C. Moreover let C^+ and C^- be the set of all nonnegative elements and the set of all nonpositive elements of C, respectively. Then C^+ and C^- are convex subsemigroups of S and*

$$C^+ \cup C^- = C, \qquad C^+ \cap C^- = \{e\}.$$

Moreover, for every $x \in C^+$ and $y \in C^-$, we have $x \leqq y$.

Let C be a nonperiodic $\mathcal{A}$-class of S. Then either every element of C is positive or every element of C is negative. In the former case, C is called a *positive nonperiodic $\mathcal{A}$-class* and, in the latter case, C is called a *negative nonperiodic $\mathcal{A}$-class.*

Theorem 6. *Let C be a positive nonperiodic $\mathcal{A}$-class of S. Then $x < xy$ and $x < yx$ for every $x, y \in C$.*

Theorem 7. *The archimedean equivalence $\mathcal{A}$ on S is the least equivalence relation $\mathcal{B}$ on S such that each $\mathcal{B}$-class is a convex subsemigroup of S.*

References

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[3] ——: The archimedean property in an ordered semigroup. J. Austral. Math. Soc., **8**, 547–556 (1968).