15. *Remark on the $A^p(G)$-algebras*[*)]

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1. Introduction. Let G denote a locally compact abelian topological group with character group $\hat{G}$, and dx (respect $d\hat{x}$) expresses the integration over G (resp. $\hat{G}$) with respect to the Haar measure. For $1 \leq p < \infty$, $A^p(G)$ denotes the linear space of all complex-valued functions in $L^1(G)$ whose Fourier transforms are in $L^p(\hat{G})$. As the linear space $A^p(G)$ is normed by $\|f\|^p = \|f\|_1 + \|\hat{f}\|_p$, then $A^p(G)$ is a semi-simple commutative Banach algebra under convolution as multiplication (see Larsen, Liu and Wang [2]). In this note, we shall show that it is regular and that some local properties hold in it (cf. Rudin [5], section 2.6). It is also proved that the abstract Silov's theorem (see Loomis [4] p. 86) holds for $A^p(G)$. The standard proof of this theorem in $L^1(G)$ (cf. Loomis [4] p. 151) seems to depend upon the uniform boundedness of the approximate identity. The author proved that the approximate identity exists for $A^p(G)$ but uniformly bounded in general (see Lai [3]). However a similar proof is obtained despite of the fact that the approximate identity in $A^p(G)$ is unbounded.

2. Closed ideals and locally properties in the algebra $A^p(G)$.

Since $A^p(G)$ has an approximate identity in the sense of Theorem 1 in Lai [3], the following proposition is immediately.

Proposition 1. *The set J of all functions of $A^p(G)$ such that the Fourier transforms have compact supports in $\hat{G}$ is a dense ideal in $A^p(G)$ with respect to A^p-topology.*

The following theorem proved for $L^1(G)$ in Loomis [4: Theorem 31 F]

Theorem 2. *A closed subset I of $A^p(G)$ is an ideal if and only if it is a translation invariant subspace.*

Proof. The necessity is immediate since $A^p(G)$ has approximate identity and the translation operator is a multiplier.

For the sufficiency, we suppose that I is a closed translation invariant subspace and consider the mapping $f \to (f, \hat{f})$ of $A^p(G)$ in $L^1(G) \times L^p(G)$, so that each continuous linear functional of $A^p(G)$ may

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be expressed in the form

$$F(f)=\int_G f(x)g(x)dx+\int_{\hat{G}}\hat{f}(\hat{x})\varphi(\hat{x})d\hat{x}$$
$$=\langle f, g\rangle+\langle\hat{f}, \varphi\rangle$$

for some pair $(g, \varphi)\in L^\infty(G)\times L^q(\hat{G})$, where $1/p+1/q=1$.

Let $I_p=\{(g, \varphi)\in L^\infty(G)\times L^q(\hat{G})\,;\, F(f)=\langle f, g\rangle+\langle f, \varphi\rangle=0$, for any $f\in I\}$. Since I is closed, $(I^\perp)^\perp=I$ (cf. Loomis [4; 8c]). For $h\in A^p(G)$, $f\in I$ and $(g, \varphi)\in I_p=I^\perp$, we have

$$F(h*f)=\int_G h*f(x)g(x)dx+\int_{\hat{G}}\widehat{h*f}(\hat{x})\varphi(\hat{x})d\hat{x}$$
$$=\int_G h(y)dy\left(\int_G \rho_y f(x)g(x)dx+\int_{\hat{G}}\widehat{\rho_y f}(\hat{x})\varphi(\hat{x})d\hat{x}\right)$$
$$=0$$

since I is translation invariant subspace, $f\in I$ implies $\rho_y f\in I$, where $\rho_y f(x)=f(x-y)$. Therefore $h*f\in I$, this shows that the closed subspace I is an ideal of $A^p(G)$. Q.E.D.

The following theorem is similar to the Theorem 2.6.2 in Rudin [5] which is proved for $L^1(G)$.

Theorem 3. *Let K be any compact set in G containing* 0 *and U be any open neighborhood of K. Then there exists a function $f\in A^p(G)$ such that $\hat{f}=1$ on K, $\hat{f}=0$ outside U and $0\le\hat{f}\le 1$.*

Proof. Let V be a symmetric compact neighborhood of the origin in $\hat{G}$ so that U contains $K+V+V$ and g, h be functions in $L^2(G)$ such that $\hat{g}$ and $\hat{h}$ are the characteristic functions of V and $K+V$ respectively. Define

$$k(x)=\frac{g(x)h(x)}{m(V)} \qquad x\in G$$

where $m(V)$ is the Haar measure of V. It is then clear that the function $k\in A^p(G)$ is desired. Q.E.D.

Remark. By translation, this theorem holds for any compact set K in $\hat{G}$ and any open neighborhood U of K.

The following theorem is essential in later.

Theorem 4. *Suppose that $f\in A^p(G)$ with $\hat{f}(0)=0$ and that $\{U_\lambda\}$ is a neighborhood system of* 0 *in G with measure less than or equal to* 1, *then given any $\varepsilon>0$, there is a net $\{k_\lambda\}$ in $A^p(G)$ such that*

(i) $\|k_\lambda\|^p<3$,

(ii) $\hat{k}_\lambda=1$ *on some neighborhood of* 0 *in U_λ and $\hat{k}_\lambda=0$ outside U_λ,*

(iii) $\|f*k_\lambda\|^p<\varepsilon$.

Proof. For $f\in A^p(G)$ and $\hat{f}(0)=0=\int_G f(x)dx$, there is a neighborhood U_λ of 0 in $\hat{G}$ such that

$$\left(\int_{U_\lambda}|\hat{f}(\hat{x})|^p d\hat{x}\right)^{1/p}<\varepsilon/2.$$

Put

$$\delta=\frac{\varepsilon}{8(1+\|f\|_1)}.$$

There is a compact set E in G such that

$$\int_{E'}|f(x)|\,dx<\delta,$$

where E' is the complement of E in G. We can find a compact set $K_\lambda\ni 0$ and a symmetric compact neighborhood V_λ in $\hat{G}$ subject to the same places of K and V in Theorem 3. Furthermore they satisfy the following conditions

1 0 is an interior point of K_λ

2 $m(K_\lambda+V_\lambda)<4m(V_\lambda)$

3 The neighborhood $U_\lambda\supset K_\lambda+V_\lambda+V_\lambda$

4 $|1-(x,\hat{x})|<\delta$ whenever $x\in E$ and $\hat{x}\in U_\lambda$.

Let g_λ and h_λ be functions in $L^2(G)$ such that g_λ and h_λ are the characteristic functions of V_λ and $K_\lambda+V_\lambda$ respectively. Define

$$k_\lambda(x)=\frac{g_\lambda(x)h_\lambda(x)}{m(V_\lambda)}\qquad(x\in G).$$

Then $k\in A^p(G)$ with $\hat{k}_\lambda=1$ on K_λ and $\hat{k}_\lambda=0$ outside U_λ, proves (ii).

Since $\hat{g}_\lambda*\hat{h}_\lambda\in C_c\subset L^p$,

$$\|\hat{k}_\lambda\|_p=\frac{1}{m(V_\lambda)}\|\hat{g}_\lambda*\hat{h}_\lambda\|_p\leq\frac{1}{m(V_\lambda)}\|\hat{g}_\lambda\|_i\|h_\lambda\|_p$$
$$=[m(V_\lambda+K_\lambda)]^{1/p}<1,$$

thus $\|\hat{k}_\lambda\|_p<1$. And

$$\|k_\lambda\|_1=\frac{1}{m(V_\lambda)}\int_G|g_\lambda(x)h_\lambda(x)|\,dx\leq\frac{1}{m(V_\lambda)}\|g_\lambda\|_2\|h_\lambda\|_2<2,$$

hence $\|k_\lambda\|^p<3$, proves (i).

Next, by $\hat{f}(0)=0=\int_G f(x)dx$, we see that

$$f*k_\lambda(x)=\int_G f(y)(k_\lambda(x-y)-k_\lambda(x))dy,$$

and

$$\|f*k_\lambda\|^p=\|f*k_\lambda\|_1+\|\hat{f}\hat{k}_\lambda\|_p.$$

It is not difficult to show that

$$\|f*k_\lambda\|_1<4\delta(1+\|f\|_1)<\varepsilon/2.$$

On the other hand,

$$\|\hat{f}\hat{k}_\lambda\|_p^p=\left(\int_{\hat{G}}|\hat{f}(\hat{x})\hat{k}_\lambda(\hat{x})|^p dx\right)=\int_{U_\lambda}+\int_{U_\lambda'}.$$

The integral over U_λ is less than

$$\sup_{\hat{x}\in U_\lambda}|\hat{k}_\lambda(\hat{x}')|^p(\varepsilon/2)^p<(\varepsilon/2)^p$$

and the integral over the complement U_λ' of U_λ is zero. Hence

$$\|\hat{f}\hat{k}_\lambda\|_p<\varepsilon/2.$$

Therefore

$$\|f*k_\lambda\|^p<\varepsilon,$$

proves (iii). Q.E.D.

Remark. By translation, this theorem holds for the case of $\hat{f}(\hat{x}_0)=0$ for some $\hat{x}_0\in\hat{G}$ in which $\{U_\lambda\}$ is a neighborhood system of $\hat{x}_0$ in $\hat{G}$.

Corollary 5. *For any $\varepsilon>0$, and $y\in E$ (compact set in G) then there is a function k_λ in $A^p(G)$ on which the Fourier transform has compact support such that*

$$\|\rho_y k_\lambda-k_\lambda\|^p<\varepsilon.$$

Proof. Choose k_λ in the net $\{k_\lambda\}$ of Theorem 4, then one can show immediately.

The following theorem is important for the later proof of Silov's theorem for the algebra $A^p(G)$ (cf. Theorem 2.6.4 of Rudin [5]).

Theorem 6. *Suppose that $f\in A^p(G)$ such that $\hat{f}(0)=0$, then there exists a net $\{v_\alpha\}\subset A^p(G)$ with $\hat{v}_\alpha=0$ in a neighborhood of 0 in $\hat{G}$ and such that*

$$\lim_\alpha \|f*v_\alpha-f\|^p=0.$$

Proof. Let $\{e_\beta\}$ be an approximate identity for $A^p(G)$ in the sense of Lai [3]. Suppose that the net $\{k_\lambda\}$ is constructed as in Theorem 4. Define

$$v_\alpha=e_\beta-k*e_\beta,\ \alpha \text{ is the ordered pair } (\beta,\lambda).$$

Evidently $v_\alpha\in A^p(G)$ and the set $\{v_\alpha\}$ may be directed by

$$(\beta_1,\lambda_1)=\alpha_1>\alpha_2=(\beta_2,\lambda_2) \text{ if and only if } \beta_1>\beta_2 \text{ and } \lambda_1>\lambda_2.$$

Then $\hat{v}_\alpha=\hat{e}_\beta(1-\hat{k}_\lambda)=0$ on some compact neighborhood of 0 in $\hat{G}$ since $\hat{k}_\lambda=1$ on some compact set containing the origin 0 as interior point,

$$\begin{aligned}\|v_\alpha*f-f\|^p&=\|e_\beta*f-k*e_\beta*f-f\|^p\\&\le\|e_\beta*f-f\|^p+\|k*(e_\beta*f)\|^p.\end{aligned}$$

Since $\lim_\beta\|e_\beta*f-f\|^p=0$ and $\lim_\lambda\|k_\lambda*(e_\beta*f)\|^p=0$ (by Theorem 4),

$$\lim_\alpha\|v_\alpha*f-f\|^p=0. \qquad \text{Q.E.D.}$$

3. Silov's theorem for $A^p(G)$. Let $\mathfrak{M}$ be the set of all regular maximal ideals of a commutative Banach algebra A. The set Δ of all continuous homomorphism of A into the complex number field is a subset of the conjugate space A^* of A and Δ is a locally compact space in the weak*-topology of A^*. The set Δ can be identified with $\mathfrak{M}$. The set of all regular maximal ideals M which contains an ideal I is called the hull of I, i.e. the hull $h(I)=\{M\in\mathfrak{M};M\supset I\}$. If E is any subset in $\mathfrak{M}$, the kernel $k(E)=\{f\in A;\hat{f}(M)=0 \text{ for all } M\in E\}=\bigcap_{M\in E}M$, which is an ideal of elements $f\in A$ such that $\hat{f}=0$ on E. If the closure of E in $\mathfrak{M}$ is defined as $h(k(E))$, then the closure can be to introduce a topology $\mathfrak{I}_{hk}$ in the space of $\mathfrak{M}$. In general, $\mathfrak{I}_{hk}$ is weaker than the weak*-topology $\mathfrak{J}_w$. As this topology $\mathfrak{J}_{hk}$ coincides with the weak*-topology $\mathfrak{J}_w$ on $\mathfrak{M}$, then the algebra A is called regular. Silove proved the following (cf. Loomis [4] p. 86, p. 151)

Theorem. *Let A be a regular semi-simple commutative Banach algebra satisfying the condition* D *and let I be a closed ideal of A. Then I contains every element f in $k(h(I))$ such that the intersection of the boundary of hull (f) with hull (I) includes no non-zero perfect set.*

Here we say the algebra A satisfying the Ditkin's condition (simply, say the condition D) if for any $f \in M \in \mathfrak{M}$, there exists a sequence $\{f_n\}$ in A such that $\hat{f}_n = 0$ in a neighborhood V_n of M and $\lim ff_n = f$ in A. If $\mathfrak{M}$ is not compact the condition D must be also satisfied for the point at infinity, i.e. for any $f \in A$, there exists a sequence $\{f_n\}$ in A such that $\{\hat{f}_n\} \subset C_c(\mathfrak{M})$ with $\lim ff_n = f$ in A.

We shall show that $A^p(G)$ is regular and satisfies the condition D and hence Silov's theorem holds for $A^p(G)$. It is known that $A^p(G)$ is a semi-simple Banach algebra, the regular maximal ideal space $\mathfrak{M}$ can be identified with the character group $\hat{G}$. For any $\hat{x} \in \hat{G}$, there corresponds a regular maximal ideal $M_{\hat{x}} \in \mathfrak{M}$ by

$$M_{\hat{x}} = \{f \in A^p(G)\,;\, \hat{x}(f) = 0 = \hat{f}(\hat{x})\} = \hat{x}^{-1}(0).$$

Theorem 7. *The algebra $A^p(G)$ is regular.*

Proof. It sufficies to show that for any closed subset $F \subset \hat{G}$ and any point $\hat{x}_0 \notin F$, there exists a function $f \in A^p(G)$ such that

$$\hat{f} = 0 \text{ on } F \text{ and } \hat{f}(\hat{x}) = 0$$

(cf. Loomis [4] p. 57). Let $U = \hat{G} - F$. Then U is an open set and $\hat{x}_0 \in U$. Choose a compact neighborhood K of $\hat{x}_0$ such that $K \subset U$. By Theorem 3 (Remark), there exists a function $k \in A^p(G)$ such that

$$\hat{k} = 1 \text{ on } K \text{ and } \hat{k} = 0 \text{ outside } U.$$

Therefore $A^p(G)$ is regular. Q.E.D.

Lemma 8. *$A^p(G)$ satisfies the condition* D *at every point $\hat{x}$ in $\hat{G}$.*

Proof. This Lemma follows from Theorem 6. That is $A^p(G)$ satisfies the condition D at the origin of G, then it holds for the points upon translation.

Lemma 9. *The algebra $A^p(G)$ satisfies the condition* D *for the point at infinity (cf. Loomis* [4] *p.* 149 *Lemma).*

Proof. This Lemma holds only for the case of non-discrete group G. The proof is similar to the case of $L^1(G)$ except the case of bounded approximate identity in $L^1(G)$.

As G is non-discrete, $\hat{G}$ is not compact. By Proposition 1, for any $f \in A^p(G)$, there exists a sequence $\{v_n\}$ in J such that

$$\lim_n f * v_n = f \qquad \text{in } A^p(G). \qquad \text{G.E.D.}$$

By Lemmas 8, 9 and Theorem 7, we see immediately that the

Silov's theorem is valid for $A^p(G)$. We restate the theorem as following (4 p. 151).

Theorem 10. *Let I be a closed ideal in $A^p(G)$ and $f \in A^p(G)$ such that $f \in k(h(I))$. Suppose furthermore that the intersection of the Silov's boundary hull (f) and hull (I) contains only the set of isolated points. Then $f \in I$.*

Corollary 11. *If I is a closed ideal in $A^p(G)$ whose hull is discrete, then $I = k(h(I))$.*

References

[1] E. Hewitt and K. A. Ross: Abstract Harmonic Analysis. I. Springer-Verlag Berlin Gottingen Heit (1963).

[2] R. Larsen, T-S Liu, and J-K Wang: On functions with Fourier transforms in L^p. Michigan Math. J., **11**, 369–378 (1964).

[3] H-C Lai: On some properties of $A^p(G)$-algebras (to appear).

[4] L. H. Loomis: An Introduction to Abstract Harmonic Analysis. Van Nostrand, New York (1953).

[5] W. Rudin: Fourier Analysis on Groups. Interscience Publishers, New York (1962).

[6] C. R. Warner: Closed ideals in the group algebra $L^1 L^2(G)$. Trans. Amer. Math. Soc., **121**, 408–423 (1966).