14. *Some Cross Norms which are not Uniformly Cross*

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It is known that all C^*-norms in the algebraic tensor product of two C^*-algebras are cross. We shall show that no C^*-norms are uniformly cross in R. Schatten's sense [4] in the algebraic tensor product of two non-abelian C^*-algebras if one of them has an anti-*-automorphism of period two. Also, some examples will show that actually there are *not uniformly cross* C^*-norms. This fact may be felt strange at a glance and will be worth researching.

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1. Preliminaries. Let E and F be Banach spaces, $E\odot F$ the algebraic tensor product of E and F, $\|\ \ \|_\beta$ a norm in $E\odot F$ and $E\hat{\otimes}_\beta F$ the tensor product of E and F with respect to $\|\ \ \|_\beta$, that is, the completion of $E\odot F$ with respect to $\|\ \ \|_\beta$.

If $\|\ \ \|_\beta$ satisfies the relation

$$\|u\otimes v\|_\beta=\|u\|\,\|v\| \text{ for each } u\in E \text{ and } v\in F,$$

then it is said to be cross; also, if $\|\ \ \|_\beta$ is cross and if for each pair of bounded linear operators ρ on E and σ on F, the relation

$$\|\Sigma_i\rho(u_i)\otimes\sigma(v_i)\|_\beta\leqq\|\rho\|\,\|\sigma\|\,\|\Sigma_iu_i\otimes v_i\|_\beta \quad \text{for each } \Sigma_iu_i\otimes v_i\in E\odot F$$

is satisfied, in other words, the operator norm of the linear operator

$$(\rho\otimes\sigma)(\Sigma_iu_i\otimes v_i)=\Sigma_i\rho(u_i)\otimes\sigma(v_i)$$

on $E\odot F$ is finite and not greater than $\|\rho\|\,\|\sigma\|$, then $\|\ \ \|_\beta$ is said to be uniformly cross (see [4], V and VI in pp. 28–29).

Let A and B be C^*-algebras. A norm $\|\ \ \|_\beta$ in the algebraic tensor product $A\odot B$ of A and B is called a C^*-norm if $\|t^*t\|_\beta=\|t\|_\beta^2$ for all $t\in A\odot B$. It is obvious that if $\|\ \ \|_\beta$ is a C^*-norm then $A\hat{\otimes}_\beta B$ becomes a C^*-algebra in the usual way.

The most natural C^*-norm in $A\odot B$ is the α-norm $\|\ \ \|_\alpha$ defined by

$$\|\Sigma_ia_i\otimes b_i\|_\alpha=\|\Sigma_i\pi_1(a_i)\otimes\pi_2(b_i)\| \qquad \text{for } \Sigma_ia_i\otimes b_i\in A\odot B,$$

using arbitrarily chosen faithful *-representations π_1 of A and π_2 of B, where the right side means the operator norm of the operator $\Sigma_i\pi_1(a_i)\otimes\pi_2(b_i)$ on the tensor product $H_1\otimes H_2$ of the representation Hilbert spaces H_1 of π_1 and H_2 of π_2 (see [6], [7]). Another C^*-norm in $A\odot B$ is referred to [1] and [3].

The reason why a C^*-norm $\|\ \ \|_\beta$ is cross lies in the facts that the α-norm is cross, that $\|t\|_\alpha\leqq\|t\|_\beta$ (Theorem 2 in [5]) and that $\|x\otimes y\|_\beta$

$\leqq \|x\| \|y\|$ (Theorem 1 in [3]).

2. A theorem.

Theorem. *Let A and B be non-abelian C^*-algebras with identities, $\| \quad \|_\beta$ a C^*-norm in $A \odot B$, π_1 a *-automorphism of A and π_2 an anti-*-automorphism of B of each period two. Then, the operator norm $\|\pi_1 \otimes \pi_2\|_\beta$ of the operator $\pi_1 \otimes \pi_2$ in $A \odot B$ with respect to $\| \quad \|_\beta$ is greater than* 1.

Proof. If $\pi_1 \otimes \pi_2$ is bounded with respect to $\| \quad \|_\beta$, it can be extended to a bounded linear operator on $A \hat{\otimes}_\beta B$ which is denoted by $\pi_1 \otimes \pi_2$ again. Since $\pi_1 \otimes \pi_2$ fixes the identity of $A \hat{\otimes}_\beta B$, $\|\pi_1 \otimes \pi_2\|_\beta \geqq 1$.

Suppose that $\|\pi_1 \otimes \pi_2\|_\beta = 1$. Then, $\pi_1 \otimes \pi_2$ is an isometry with respect to $\| \quad \|_\beta$ because $(\pi_1 \otimes \pi_2)^{-1} = \pi_1^{-1} \otimes \pi_2^{-1} = \pi_1 \otimes \pi_2$ on $A \odot B$. Thus, by Kadison's theorem [2], we know that $\pi_1 \otimes \pi_2$ is a C^*-homomorphism, that is, an adjoint- and square-preserving linear operator. On the other hand, there are elements $x_1, x_2 \in A$ and $y_1, y_2 \in B$ such that $x_1 x_2 \neq x_2 x_1$ and $y_1 y_2 \neq y_2 y_1$. Put here $t = x_1 \otimes y_1 + x_2 \otimes y_2$. Then a simple computation shows that

$$(\pi_1 \otimes \pi_2)(t^2) \neq ((\pi_1 \otimes \pi_2)(t))^2,$$

a contradiction. Therefore the proof is completed.

Directly from the above theorem, we know that if A and B are non-abelian C^*-algebras with identities and if B has an anti-*-automorphism of period two then no C^*-norms in $A \odot B$ are uniformly cross. In fact, denoting by ι the identity *-automorphism of A and by φ an anti-*-automorphism of B in the assumption, we have

$$\|\iota \otimes \varphi\|_\beta > 1 = \|\iota\| \|\varphi\|.$$

3. Examples. In the following, we shall consider some examples. Let H be a Hilbert space, $(\xi_\mu)_{\mu \in I}$ a complete orthonormal system of H, $B(H)$ the C^*-algebra of bounded linear operators on H which are regarded as matrices with respect to (ξ_μ), τ the transposition of $B(H)$ with respect to (ξ_μ):

$$y = (\lambda_{\mu\nu}) \to {}^t y = (\lambda_{\nu\mu}).$$

It is easy to see that τ is an anti-*-automorphism of $B(H)$ of period two. Thus we know, from the theorem mentioned above, that if A is a non-abelian C^*-algebra with an identity ι the identity *-automorphism of A and $\| \quad \|_\beta$ a C^*-norm in $A \odot B(H)$ then $\|\iota \otimes \tau\|_\beta > 1$ and no C^*-norms in $A \odot B(H)$ are uniformly cross.

We want to proceed to consider the operator $\iota \otimes \tau$. Suppose moreover that A is acting on a Hilbert space K, then, $A \odot B(H)$ becomes a sub-algebra of the von Neumann algebra tensor product $B(K) \otimes B(H)$ and the α-norm in $A \odot B(H)$ coincides with the operator norm $\| \quad \|$. By an easy computation, it turns out that the operator $\iota \otimes \tau$ is nothing but *a generalized transposition* of $A \odot B(H)$ with respect to (ξ_μ):

$$(x_{\mu\nu}) \to (x_{\nu\mu}).$$

Next, let $A=B(K)$ and let H and K are infinite-dimensional. Then it is concluded that $\iota\otimes\tau$ in $A\odot B(H)$ is unbounded with respect to the α-norm, that is, that $\|\iota\otimes\tau\|_\alpha=\infty$ happens.

The reason is as following. Let $(\eta_\kappa)_{\kappa\in J}$ be a complete orthonormal system of K and $B(K)$ be represented as the matrix algebra with respect to (η_κ). We may assume that the sets I and J contain the sequence of positive integers. When we put

$$x^{(k)}=\begin{bmatrix} 0 & 0\cdots 0 & \overset{k}{\breve{1}} & 0\cdots \\ & & & \\ & & 0 & \\ & & & \end{bmatrix}, \qquad k=1,2,\cdots,$$

and

$$t^{(n)}=\begin{bmatrix} x^{(1)} & & \\ x^{(2)} & & \\ \vdots & & \\ x^{(n)} & 0 & \\ 0 & & \\ 0 & & \\ \vdots & & \end{bmatrix}, \qquad n=1,2,\cdots,$$

then $\{t^{(n)}\}\subset A\odot B(H)$ and $\|t^{(n)}\|_\alpha=1$, while $\|(\iota\otimes\tau)(t^{(n)})\|_\alpha=\sqrt{n}$, $n=1,2,\cdots$. Therefore $\|\iota\otimes\tau\|_\alpha=\infty$.

At last several remarks we give. (a) If A is an abelian C^*-algebra, then the α-norm is the only C^*-norm in $A\odot B(H)$ (see [5]) and $\iota\otimes\tau$ in $A\odot B(H)$ can be extended to an anti-*-automorphism on the tensor product $A\hat{\otimes}_\alpha B(H)$ of course with $\|\iota\otimes\tau\|_\alpha=1$.

(b) If A is a non-abelian C^*-algebra with an identity and if H is finite-dimensional, then the α-norm is the only C^*-norm also ([5]), $A\odot B(H)$ is complete with respect to it and $\infty>\|\iota\otimes\tau\|_\alpha>1$.

(c) If a C^*-algebra A contains a family of infinite number of equivalent projections which are mutually orthogonal, and if H is infinite-dimensional, then $\iota\otimes\tau$ on $A\odot B(H)$ is unbounded with respect to the α-norm. As such A we can consider for example a C^*-algebra which contains a factor of type I_∞. The proof is an easy modefication of the above discussion.

References

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