13. *On Vector Measures. I*

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(Comm. by Kinjirô KUNUGI, M. J. A., Jan. 12, 1970)

1. Introduction. In [1] Dinculeanu and Kluvanek have proved the following result.

Let S be a set, R a tribe (σ-ring) of subsets of S, X a locally convex linear space with topology defined by a family $\{\| \ \|_p\}_{p \in P}$ of semi-norms, and $m ; R \to X$ a vector measure. Then for every $p \in P$ there exists a finite non-negative measure ν_p on R such that

(1) $\lim_{\nu_p(A) \to 0} \|m(A)\|_p = 0$

(2) $\nu_p(E) \leqq \sup \{\|m(A)\|_p ; A \subset E, A \in R\}$ ([1] Theorem 1)

They also raised the following problem: whether this theorem remains valid if the tribe is replaced by a semi-tribe (δ-ring)? In this paper we shall give the negative answer for the problem. And in case R is a semi-tribe we shall show that the above theorem remains true under a weaker property than (1) and property (2). (cf. Theorem 1)

In this paper we suppose that X is a normed space in order to simplify the proof.

2. Vector measures. Definition 1. Let S be a set. A nonvoid class R of subsets of S is called a semi-tribe (δ-ring) if;

(1) $A, B \in R \Rightarrow A \cup B \in R$, $A - B \in R$.

(2) $A_n \in R \ (n=1, 2, \cdots) \Rightarrow \bigcap_{n=1}^{\infty} A_n \in R$.

From this definition it follows that a semi-tribe R has the following properties.

(3) $A_n \in R$, $A \in R$ and $A_n \subset A (n=1, 2, \cdots) \Rightarrow \bigcup_{n=1}^{\infty} A_n \in R$

(4) if we set $R_A = \{B \cap A ; B \in R\}$ for any $A \in R$, then R_A is a tribe on A.

Suppose that X is a normed space and $\tilde{X}$ its completion.

Definition 2. Let R be a clan (ring). A set function m defined on R with values in X is called a vector measure if the following conditions are satisfied

(1) $m(\emptyset) = 0$

(2) for every sequence $\{E_n\}$ of mutually disjoint sets of R such that $E = \bigcup_{n=1}^{\infty} E_n \in R$, $m(E) = \sum_{n=1}^{\infty} m(E_n)$.

For every $E \in R$, we set $\tilde{m}(E) = \sup \{\|m(A)\| ; A \subset E, A \in R\}$. Then it is easy to see that $\tilde{m}$ is increase, subadditive on R.

Lemma 1. *If R is a semi-tribe, $\tilde{m}$ has the following properties.*

(1) $0 \leqq \tilde{m}(E) < +\infty$ *for every* $E \in R$.

(2) *for every decreasing sequence* $\{E_n\}$ *of sets of* R *such that* $\lim_{n\to\infty} E_n = \emptyset$, *we have* $\lim_{n\to\infty} \tilde{m}(E_n) = 0$.

These results was essentially proved by Gould ([3], Theorem 2.6 and Theorem 3.6).

Theorem 1. *Let R be a semi-tribe, X a normed space and $m; R \to X$ a vector measure. Then there exists a finite non-negative measure ν on R such that*

(1) *for any $A \in R$ and any number $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon, A) > 0$ such that $B \in R$, $B \subset A$ and $\nu(B) < \delta \Rightarrow \|m(B)\| < \varepsilon$.*

(2) $\nu(E) \leqq \tilde{m}(E) = \sup\{\|m(A)\|; A \subset E, A \in R\}$ *for every* $E \in R$.

Proof. For any $A \in R$ if m_A is the restriction of m to R_A, then from Dinculeanu and Kluvanek ([1] Theorem 1) there exists a finite non-negative measure ν_A on R_A such that

(i) for every number $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon, A) > 0$ such that if $B \in R, B \subset A$ and $\nu_A(B) < \delta$ then $\|m_A(B)\| = \|m(B)\| < \varepsilon$ and

(ii) $\nu_A(B) = \sup\{\|m_A(C)\|; C \subset B, C \in R_A\}$
$= \sup\{\|m(C)\|; C \subset B, C \in R\} = \tilde{m}(B)$

Now we set $\nu(E) = \sup\{\nu_A(E \cap A); A \in R\}$ for every $E \in R$. Then we can prove that ν has the following properties.

(a) $0 \leqq \nu(E) \leqq \tilde{m}(E)$ for every $E \in R$: this is immediate by (ii).

(b) ν is a measure on R.

(b_1) We easily see that $\nu(\emptyset) = 0$.

(b_2) ν is finitely additive: If $E, F \in R$ and $E \cap F = \emptyset$, we have $\nu(E \cup F) \leqq \nu(E) + \nu(F)$. Therefore we have only to prove the inverse inequality. For any $\varepsilon > 0$ there exists a $A \in R$ such that $\nu_A(E \cap A) > \nu(E) - \frac{1}{2}\varepsilon$ and $\nu_A(F \cap A) > \nu(F) - \frac{1}{2}\varepsilon$. Hence $\nu(E) + \nu(F) - \varepsilon < \nu_A(E \cap A) + \nu_A(F \cap A) = \nu_A((E \cup F) \cap A) \leqq \nu(E \cup F)$. Since $\varepsilon > 0$ is arbitrary, we have $\nu(E) + \nu(F) \leqq \nu(E \cup F)$. Thus ν is finitely additive.

(b_3) ν is countably additive: Let $\{E_n\}$ be a decreasing sequence of sets of R such that $\lim_{n\to\infty} E_n = \emptyset$.

By (a) we have $0 \leqq \nu(E_n) \leqq \tilde{m}(E_n)$ for all n, and by Lemma 1 (2) $\lim_{n\to\infty} \tilde{m}(E_n) = 0$.

Hence

$$\lim_{n\to\infty} \nu(E_n) = 0.$$

Thus ν is countably additive i.e. ν is a measure. It is easy to verify the property (1). Q.E.D.

The property (1) of Theorem 1 cannot be replaced by the stronger condition $\lim_{\nu(A)\to 0} \|m(A)\| = 0$. The following example is a negative answer of the problem posed by Dinculeanu and Kluvanek [1].

Counter example. Let S be an infinite set, R the semi-tribe of all finite sets in S, X the Banach space of bounded functions on S with sup norm and φ_A the characteristic function of A. Then

$$\varphi_A \in X \quad \text{and} \quad \|\varphi_A\|=1$$

Define $m; R \to X$ by $m(A)=\varphi_A$ if $A \in R$, $A \neq \emptyset$, and $m(\emptyset)=0$. Then m is a vector measure. If $\lim_{\nu(A)\to 0} \|m(A)\|=0$ and (2) is satisfied for some non-negative measure ν on R, we have $\nu(\{s\})>0$ for every point $s \in S$. As we have $\nu(A) \leqq 1$ for every $A \in R$, there exists a sequence $\{s_n\}$ of points of S such that $\nu(\{s_n\})<\frac{1}{n}$ for $n=1, 2, \cdots$, so $\lim_{n\to\infty} \|m(\{s\})\|=0$. But $\|m(\{s_n\})\|=\|\varphi_{\{s_n\}}\|=1$ for $n=1, 2, \cdots$. This contradiction shows that R has not the property $\lim_{\nu(A)\to 0} \|m(A)\|=0$.

Remarks. (1) In Theorem 1, the semi-tribe cannot be replaced by the clan (see Dinculeanu and Kluvanek [1] example).

(2) The condition (1) of Theorem 1 is equivalent to the following.

(1′) $\nu(E)=0$, $E \in R \Rightarrow m(E)=0$

This is due to Neumann ([4] Theorem 11.2.4).

Theorem 2. *Let R be a clan, φ the semi-tribe generated by R. A vector measure $m; R \to X$ can be extended to a vector measure $m; \varphi \to \tilde{X}$ if and only if there exists a finite non-negative measure ν on R such that*

$$\nu(E)=0, E \in R \Rightarrow m(E)=0. \qquad \text{([1] Theorem 2, Corollary 2)}$$

Proof. The necessity is clear by Theorem 1. The sufficiency is due to Dinculeanu and Kluvanek ([1], Theorem 2, Corollary 2).

Q.E.D.

References

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