12. *Ergodic and Mixing Properties of Measure Preserving Transformations*

By Eijun KIN
Department of Mathematics, Tokyo Metropolitan University, Tokyo

(Comm. by Kinjirô KUNUGI, M. J. A., Jan. 12, 1970)

Totoki [6] has shown that strongly mixing Gaussian flows are all order mixing. As is well-known, the all order mixing implies the weak mixing and the weak mixing implies the ergodicity. Conversely, one can ask for which class of transformations ergodicity implies all order mixing. Halmos [2] has proved that if a continuous automorphism of a compact Abelian group is ergodic, then the automorphism is strongly mixing (i.e. 1-order mixing), and Rohlin [4] has proved further that every ergodic continuous automorphism of a compact Abelian group is all order mixing.

In this paper we study some classes of the transformations of which ergodicity and strong mixing imply all order mixing respectively. Our transformations were first topologically studied by Keynes and Robertson in their paper [1].

Let $(\Omega, \mathscr{B}, m)$ be a probability measure space and I be the set of all integers or real numbers. Consider a group G of homeomorphisms of I and for each $g \in G$, define an automorphism T_g of $(\bigotimes_{i\in I} \Omega, \bigotimes_{i\in I} \mathscr{B}, \bigotimes_{i\in I} m)$ as follows:

$$T_g(\omega_i \,|\, i \in I) = (\omega_{g(i)} \,|\, i \in I) \qquad (\omega_i \,|\, i \in I) \in \bigotimes_{i\in I} \Omega.$$

We call each T_g a *G-index automorphism*.

Definitions. (i) T_g is *ergodic* if for every $E, F \in \bigotimes_{i\in I} \mathscr{B}$ with positive measure, there exists a positive integer n such that

$$\bigotimes_{i\in I} m(T_g^n E \cap F) > 0.$$

(ii) T_g is *weakly mixing* if the product automorphism $T_g \otimes T_g$ is ergodic.

(iii) T_g is *strongly mixing* if for every $E, F \in \bigotimes_{i\in I} \mathscr{B}$ with positive measure,

$$\lim_{n\to\infty} \bigotimes_{i\in I} m(T_g^n E \cap F) = \bigotimes_{i\in I} m(E) \bigotimes_{i\in I} m(F).$$

Lemma 1. *Let $g \neq e$. If T_g is ergodic, then there exists a positive integer n such that $g^n(\alpha) \cap \beta = \emptyset$ holds for every finite subsets α, β of I.*

Proof. Suppose there exist finite subsets α, β of I such that $g^n(\alpha) \cap \beta = \emptyset$ for all n. Choosing $A, B \in \mathscr{B}$ with positive measure so that $m(A\cap B)=0$ and putting

$$E=(\bigotimes_{i\in\alpha}A)\otimes(\bigotimes_{i\in I-\alpha}\Omega)\quad\text{and}\quad F=(\bigotimes_{i\in\beta}B)\otimes(\bigotimes_{i\in I-\beta}\Omega),$$

one readily obtains $\bigotimes_{i\in I}m(T_g^n E\cap F)=0$ for all n. But this contradicts the ergodicity of T_g. The proof is completed.

Theorem 2. *The following statements are equivalent to one another*:

(i) T_g *is ergodic for all* $g\neq e$,

(ii) G *has no elements of finite order except the unit element and* (I,G) *is a strongly effective group*,

(iii) T_g *is all order mixing for all* $g\neq e$.

Clearly (iii) implies (i). We shall establish the theorem in the following propositions.

Proposition 3. (i) *implies* (ii).

Proof. Suppose there is an element $g(\neq e)$ in G such that there exists a positive integer n with $g^n=e$. Let $G_0=\{e,g^{\pm 1},\cdots,g^{\pm(n-1)}\}$. Then there exist finite subsets α, β of I so that $g(\alpha)\cap\beta=\emptyset$ for all $g\in G_0$. Noticing that for every $k\geqq 0$, $g^{\pm k}=g^{\pm j}$, where $k\equiv j \bmod |n|$, one can immediately find that there exist $A,B\in\bigotimes_{i\in I}\mathscr{B}$ with positive measure such that $\bigotimes_{i\in I}m(T_g^kA\cap B)=0$ for every k and $g\in G_0$. This contradiction shows that G has no elements of finite order except e. Next suppose (I,G) is not strongly effective. Then there exist $g(\neq e)$ in G and i in I with $g(i)=i$. Moreover, $g^n\neq e$ and $g^n(i)=i$ for all $n\neq 0$. Let $\alpha=\beta=\{i\}$. Take A from $\mathscr{B}$ with $m(A)m(A^c)>0$ and put

$$\tilde{A}=(\bigotimes_{i\in\alpha}A)\otimes(\bigotimes_{i\in I-\alpha}\Omega)\quad\text{and}\quad\tilde{B}=(\bigotimes_{i\in\beta}A^c)\otimes(\bigotimes_{i\in I-\beta}\Omega).$$

Then for all n, $\bigotimes_{i\in I}m(T_g^n\tilde{A}\cap\tilde{B})=0$. This contradicts, too. The result follows.

Proposition 4. (ii) *implies* (iii).

Proof. Let r be an arbitrary positive integer. Consider a subfamily $\{k_{n,j}\}_{j=0}^r$ of integers satisfying the conditions: $k_{n,j-1}<k_{n,j}$ and $\lim_{n\to\infty}\min_{1\leqq j\leqq r}[k_{n,j}-k_{n,j-1}]=\infty$ and consider a sequence $A_0,A_1,\cdots,A_r$ of $\bigotimes_{i\in I}\mathscr{B}$-measurable sets which have positive measure. Let $g\neq e$.

Case I. Suppose that for every j, $0\leqq j\leqq r$,

$$A_j=(\bigotimes_{i\in\alpha_j}A_{i,j})\otimes(\bigotimes_{i\in I-\alpha_j}\Omega)$$

where each α_j is a finite subset of I. Set $E_{i,j}=\{n\in Z: g^n(\alpha_i)\cap\alpha_j\neq\emptyset\}$ where Z denotes the set of all integers. Then we see that $E_{i,j}, i, j=0,1,\cdots,r$, are finite subsets of Z. So there exists a positive integer $p_{i,j}$ such that for all $n>p_{i,j}$, $g^{k_{n,i}}\alpha_i\cap g^{k_{n,j}}\alpha_j=\emptyset\,(i\neq j)$. Put $p=\max_{0\leqq i\neq j\leqq r}p_{i,j}$, then $g^{k_{n,i}}\alpha_i\cap g^{k_{n,j}}\alpha_j=\emptyset$ $(0\leqq i\neq j\leqq r)$ for all $n>p$. Thus we have

$$\bigotimes_{i\in I}m\left(\bigcap_{j=0}^{r}T_g^{k_{n,j}}A_j\right)=\prod_{j=0}^{r}\bigotimes_{i\in I}m(A_j)\ (n>p).$$

It is easily verified that if each A_j is the finite union of the sets of the form in Case I, then

$$\lim_{n\to\infty} \bigotimes_{i\in I} m\left(\bigcap_{j=0}^{r} T_g^{k_{n,j}} A_j\right) = \prod_{j=0}^{r} \bigotimes_{i\in I} m(A_j).$$

Case II. Let $A_0, A_1, \cdots, A_r$ be arbitrary $\bigotimes_{i\in I}\mathcal{B}$-measurable sets. Then for any positive number ε, there exist $\bigotimes_{i\in I}\mathcal{B}$-measurable sets $B_0, B_1, \cdots, B_r$ such that each B_j is the finite union of the sets of the form in Case I and for every $k_{n,j}$

$$\bigotimes_{i\in I} m(T_g^{k_{n,j}} A_j \ominus T_g^{k_{n,j}} B_j) < \varepsilon.$$

Therefore

$$\bigotimes_{i\in I} m\left(\bigcap_{j=0}^{r} T_g^{k_{n,j}} A_j \ominus \bigcap_{j=0}^{r} T_g^{k_{n,j}} B_j\right) < (r+1)\varepsilon$$

and there exists a positive integer n_0 such that for all $n > n_0$,

$$\left|\bigotimes_{i\in I} m\left(\bigcap_{j=0}^{r} T_g^{k_{n,j}} B_j\right) - \prod_{j=0}^{r} \bigotimes_{i\in I} m(B_j)\right| < \varepsilon.$$

Thus we have

$$\left|\bigotimes_{i\in I} m\left(\bigcap_{j=0}^{r} T_g^{k_{n,j}} A_j\right) - \prod_{j=0}^{r} \bigotimes_{i\in I} m(A_j)\right| < (2r+3)\varepsilon.$$

and this completes the proof.

Corollary 5. *Let G be the group generated by a nontrivial homeomorphism g of I. Then the following statements are equivalent to one another:*

(i) *T_g is ergodic,*

(ii) *T_g is strongly mixing,*

(iii) *T_g is all order mixing.*

Remark. Let $I = Z$ and $g(i) > i$ for every $i \in I$. The G-index automorphism T_g is a Kolmogorov automorphism.

Next we shall show that the foregoing statements hold for a certain flow.

Let I be the set of all real numbers and consider a topological flow $G = \{g_t\}$ on I. Let T_{g_t} be a G-index automorphism of $(\bigotimes_{i\in I}\Omega, \bigotimes_{i\in I}\mathcal{B}, \bigotimes_{i\in I} m)$ and put $S_t = T_{g_t}$. Then $\{S_t\}$ will be called a *G-index flow* on the product measure space. We assume that the set $\{g_s(\alpha) : 0 \leqq |s| \leqq t\}$ is bounded for any bounded subset α of I and for any positive number t.

Lemma 6. *If a G-index flow $\{S_t\}$ is ergodic, then there exists a positive number t such that $g_t(\alpha) \cap \beta = \emptyset$ holds for every bounded subsets α, β of I.*

We have the following theorem whose proof is similar to that of Theorem 2.

Theorem 7. *The following statements are equivalent to one another:*

(i) *$\{S_t\}$ is ergodic,*

(ii) *It holds that* $g_t(i) \neq i$ *for all* $t \neq 0$ *and* $i \in I$,

(iii) $\{S_t\}$ *is strongly mixing,*

(iv) $\{S_t\}$ *is all order mixing.*

Remark. If for all $i \in I$ and every pair t, s $(t > s)$, $g_t(i) > g_s(i)$ then $\{S_t\}$ is a Kolmogorov flow.

The following theorem supplies a necessary and sufficient condition for weakly mixing property of a general measurable flow.

Theorem 8. *Let* $\{T_t\}$ *be a measurable flow on* $(\Omega, \mathscr{B}, m)$. $\{T_t\}$ *is weakly mixing if and only if for every pair* $A, B \in \mathscr{B}$ *with positive measure, there exists a subset* M *of* $[0, +\infty)$ *satisfying the conditions:*

(i) $\lim_{T \to \infty} \frac{L_T(M)}{T} = 0$,

(ii) $m(T_t A \cap B) > 0$ $(t \notin M)$,

where $L_T(M) = \sup_{s<T} [s : s \in M]$ $(T > 0)$.

Proof. Suppose $\{T_t\}$ is weakly mixing. Then for every $A, B \in \mathscr{B}$ with positive measure, there exists a subset M_0 of $[0, +\infty)$ with density zero such that

$$\lim_{t \to \infty, t \notin M_0} m(T_t A \cap B) = m(A)m(B) \text{ (see [3]).}$$

Thus for all t not in M_0 and larger than some positive number t_0, $m(T_t A \cap B) > 0$. Let $M = M_0 \cup [0, t_0)$ and $L_T(M) = \sup_{s<T} [s : s \in M]$ $(T > 0)$. Obviously the set M satisfies the conditions (i) and (ii). Conversely, let the set M satisfy the conditions (i) and (ii) and t be a given positive number. The $N \not\subset M$, where $N = \{kt : k = 1, 2, \cdots\}$. In fact this follows from that the upper density of N is positive. Therefore, T_t is ergodic and hence $\{T_t\}$ is weakly mixing (see [5]).

The author expresses his sincerest thanks to Prof. S. Tsurumi and Prof. Y. Ito who have given valuable advices.

References

[1] Harvey B. Keynes and James B. Robertson: On ergodicity and mixing in topological transformation groups. Duke Math. J., **35**, 4 (1968).

[2] P. R. Halmos: Lectures on ergodic theory. Math. Soc. of Japan, Tokyo (1956).

[3] E. Hopf: Ergodentheorie. Berlin (1937).

[4] V. A. Rohlin: On the entropy of an automorphism of a compact commutative group. Theory of probability and its applications, VI, 3 (1961).

[5] ——: Selected topics from the metric theory of dynamical systems. A.M.S. Translations, Ser. 2, **49**, 171–240 (1966).

[6] H. Totoki: The mixing property of Gaussian flows. Mem. Fac. Sci. Kyushu Univ., **18**, 2 (1964).