10. Characterizations of Strongly Regular Rings

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In this note by a ring we shall mean a not necessarily commutative but associative ring and by radical of the ring we mean the Jacobson radical (see N. Jacobson [5]). Following J. von Neumann [13] we shall say that the ring A is regular if, for every element a of A, there exists an element x in A such that $a=axa$. It is well known that the class of regular rings plays a very important role in the abstract algebra, in the theory of Banach algebras (cf. C. E. Rickart [15]) and in the continuous geometry (see J. von Neumann [14]). An interesting result is that the ring of all linear transformations of a vector space over a division ring is a regular ring. Some ideal-theoretical characterizations of regular rings have been obtained by L. Kovács [7] and J. Luh [11].

A ring A is called strongly regular if to every element a of A there exists at least one element x in A such that $a=a^2x$ (See R. F. Arens and I. Kaplansky [2]). It can be seen that every strongly regular is regular (see T. Kandô [6]). Following E. Hille [4] a ring A is said to be a two-sided ring if every one-sided (left or right) ideal of A is a two-sided ideal of A. Evidently every division ring and every commutative ring is a two-sided ring. It is easy to see that there exists two-sided ring which is neither commutative nor a division ring. Two-sided rings called as duo rings have formerly been investigated by E. H. Feller [3], G. Thierrin [17] and S. Lajos [8]. Thierrin using the classical method of N. H. McCoy [12] has verified that every two-sided ring can be represented as a subdirect sum of subdirectly irreducible two-sided rings.

First named author has recently obtained some ideal-theoretical characterizations of two-sided regular rings which are analogous to his characterizations of semilattices of groups (see S. Lajos [8]–[10]). S. Lajos' earlier criteria are contained in the following result which will be stated here with no proof.

Theorem. *For an associative ring A the following eleven conditions are equivalent with each other:*

(I) *A is a strongly regular ring.*

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(II) *$L\cap R=LR$ for every left ideal L and for every right ideal R of A.*

(III) *The intersection of any two left ideals is equal to their product and the same for right ideals too.*

(IV) *$L\cap I=LI$ and $R\cap I=IR$ for every left ideal L, for every right ideal R and, for every two-sided ideal I of A.*

(V) *A is a subdirect sum of division rings.*[1)]

(VI) *A is a regular ring with no nonzero nilpotent elements.*

(VII) *A is a two-sided regular ring.*

(VIII) *The intersection of any two left ideals coincides with their product.*

(IX) *The intersection of any two right ideals coincides with their product.*[2)]

(X) *$L\cap I=LI$ holds for every left ideal L and for every two-sided ideal I of A.*

(XI) *$R\cap I=IR$ holds for every right ideal R and for every two-sided ideal I of A.*

Remark. We mention a nontrivial example for a two-sided regular ring which is neither commutative nor a division ring. Let A be the direct sum of two non-commutative division rings. Then A obviously has the wished properties.

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1) F. Szász [16] proved that a ring with minimum condition on principal right ideals is a discrete direct sum of division rings if and only if it has no nonzero nilpotent elements.

2) The equivalence of the conditions (I) and (IX) was proved by V. A. Andrunakievič [1].

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