8. *Some Theorems on Cluster Sets of Set-Mappings*

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1. This is a résumé of the paper which will appear elsewhere [4]. A set-mapping F from a non-empty set A into a set B is, by definition, a mapping from A into the totality of subsets of B, so that, for every $a \in A$, $F(a)$ denotes a (possibly empty) subset of B. A non-trivial example known to complex variable analysts is, of course, a multiple-valued analytic function obtained by analytic continuation throughout a plane domain D starting with a fixed function element with the centre in D. This defines a set-mapping from D into the Riemann sphere. Now, let T and S be topological spaces and F be a set-mapping from a subset $U \neq \emptyset$ of T into S. Let $G \neq \emptyset$ be a subset of U and $t_0 \in \bar{G}$, here and elsewhere, "bar" means the closure in the considered spaces. Then the cluster set $C_G(F, t_0)$ of F at t_0 relative to G is defined by the following:

$$C_G(F, t_0) = \bigcap \overline{F(N \cap G)},$$

where the intersection is taken over all neighbourhoods N of t_0 in T with

$$F(N \cap G) = \bigcup_{t \in N \cap G} F(t).$$

If, in particular, T is the disk $|z| \leqq 1$, U is $|z| < 1$ and $e^{i\theta}$ is a point of $|z| = 1$, then the full cluster set $C_U(F, e^{i\theta})$, a curvilinear cluster set $C_\gamma(F, e^{i\theta})$, the radial cluster set $C_\rho(F, e^{i\theta})$ and an angular cluster set $C_\Delta(F, e^{i\theta})$ at $e^{i\theta}$ are the cluster sets corresponding respectively to $G = U$, a simple arc in U with the initial point in U and the terminal point $e^{i\theta}$, the radius drawn to $e^{i\theta}$ and an angular domain Δ in U with the vertex at $e^{i\theta}$.

2. **Size of cluster sets.** We consider the case where T and S are metrizable and S is compact.

Theorem 1. *Let F be an arbitrary set-mapping from a subset $U \neq \emptyset$ of T into S such that $F(t) \neq \emptyset$ for any point $t \in U$. Let $\Sigma \neq \emptyset$ be closed in S and let K be the boundary (in T) of U. We set, for every $t \in K$,*

$$f(t) = \sup \text{ (inf resp.) } \{\text{dis}(\Sigma, \alpha);\ \alpha \in C_U(F, t)\}.$$

Then f is an upper (lower resp.) semi-continuous function on K. We have the same conclusion if we replace $\text{dis}(\Sigma, \alpha)$ in the definition of f by

$$\overline{\text{dis}}(\Sigma, \alpha) = \sup\{\text{dis}(s, \alpha);\ s \in \Sigma\}.$$

3. Young's Rome theorem and Collingwood's maximality theorems for set-mappings. In the rest of this note, U denotes the open unit disk, T the closed unit disk, K the unit circle and $e^{i\theta}$ a point of K. In this section we assume that S is a metrizable space which can be covered by a countable number of compact sets. We obtain Young's theorem for set-mappings ([5], cf. [2]):

Theorem 2. *Let F be an arbitrary set-mapping from K into S. Then we have*

$$C_l(F, e^{i\theta}) = C_K(F, e^{i\theta}) = C_r(F, e^{i\theta})$$

except perhaps for a countable set of points on K. Here,

$$C_r(F, e^{i\theta}) = \bigcap_{\eta>0} \overline{F(N_\eta)}$$

with

$$N_\eta = \{e^{i\varphi} \in K;\ \theta - \eta < \varphi < \theta\};$$
$$F(N_\eta) = \bigcup_{e^{i\varphi} \in N_\eta} F(e^{i\varphi})$$

and $C_l(F, e^{i\theta})$ is defined dually.

This theorem has a

Corollary (Collingwood's symmetric maximality theorem for set-mappings, cf. [2], [3]]. *Let F be an arbitrary set-mapping from U into S. Then we have*

$$C_{BL}(F, e^{i\theta}) = C_U(F, e^{i\theta}) = C_{BR}(F, e^{i\theta})$$

except perhaps for a countable set of points on K. Here,

$$C_{BR}(F, e^{i\theta}) = \bigcap_{\eta>0} \overline{C(F, \theta - \eta < \varphi < \theta)}$$

with

$$C(F, \theta - \eta < \varphi < \theta) = \bigcup_{\theta - \eta < \varphi < \theta} C(F, e^{i\varphi})$$

and $C_{BL}(F, e^{i\theta})$ is defined dually.

Theorem 3 (Collingwood's maximality theorem for set-mappings, cf. [2], [3]). *Let $\Delta(\theta)$ be the angular domain of the vertex $e^{i\theta}$ obtained by the rotation of an angular domain $\Delta(0)$ in U having the vertex at $z=1$ and bisected by the diameter at $z=1$. Let F be an arbitrary set-mapping from U into S. Then we have*

$$C_{\Delta(\theta)}(F, e^{i\theta}) = C_U(F, e^{i\theta})$$

except perhaps for a set of first Baire category on K.

Let F be a set-mapping from U into S. Then we say that F is continuous on the circle $L_r: |z| = r$ $(0 < r < 1)$ if $F(w) \neq \emptyset$ for any $w \in L_r$ and if for any $z_0 \in L_r$ and for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\sup \{\mathrm{dis}(s, \overline{F(z)});\ s \in F(z_0)\} < \varepsilon$$

for any $z \in L_r$ with $|z - z_0| < \delta$.

Theorem 4 (Collingwood's maximality theorem for continuous set-mappings, cf. [2], [3]). *Let F be a set-mapping from U into S such that F is continuous on every circle L_r with $r_0 < r < 1$, r_0 being a constant. Then we have*

$$C_\rho(F, e^{i\theta}) = C_U(F, e^{i\theta})$$

except perhaps for a set of points of first Baire category on K.

4. Bagemihl's ambiguous point theorem for set-mappings. In this section S is a compact metrizable space.

Theorem 5 (cf. [1]). *Let F be a set-mapping from U into S such that $F(z) \neq \emptyset$ for any z, $r_0 < |z| < 1$, r_0 being a constant. Then, there exists a countable set E on K such that for every $e^{i\theta} \in K \setminus E$ and for every pair of simple arcs γ_1 and γ_2 in U at $e^{i\theta}$, we have*

$$C_{\gamma_1}(F, e^{i\theta}) \cap C_{\gamma_2}(F, e^{i\theta}) \neq \emptyset.$$

References

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