7. *Sufficient Conditions for the Normality of the Product of Two Spaces*

By Masahiko ATSUJI
Department of Mathematics, Josai University, Saitama

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In this note, an application of [2] and [3] is shown which deduces known best results giving sufficient conditions stated in the title. An example is presented which suggests the reason why we have generally had only few necessary conditions of each of X and Y separately for the normality of $X\times Y$.

Spaces in this note are normal (and so Hausdorff). We use notations and terminologies in [2] and [3].

Theorem 1. *Suppose that X is normal and $\mathfrak{m}$-paracompact, Y is normal and has an open base for closed sets of power $\leqq\mathfrak{m}$, and Y is upper compact at X, then $X\times Y$ is normal.*

Proof. Let $\{A_x\subset Y\,;\,x\in X\}$ and $\{B_x\subset Y\,;\,x\in X\}$ be arbitrary families with

$$(1)\qquad \limsup_a A_x\cap\limsup_a B_x=\emptyset$$

for any $a\in X$. Let $\{E_\lambda\,;\,\lambda\in\Lambda\}$ be an open base for closed sets in Y, including the empty set and Y, with $\|\Lambda\|\leqq\mathfrak{m}$. Put $A=\bigcup_{x\in X}(x,A_x)$ and $B=\bigcup_{x\in X}(x,B_x)$, then

$$O_\lambda=\{x\,;\,\bar{A}[x]\subset E_\lambda\}\cap\{x\,;\,\bar{B}[x]\subset\mathcal{C}\overline{E_\lambda}\}$$

is open by Proposition 3 in [3]. Since, for any $x\in X$, $\bar{A}[x]$ and $\bar{B}[x]$ are disjoint closed sets of the normal space Y, there is an E_λ such that $\bar{A}[x]\subset E_\lambda$ and $\overline{E_\lambda}\subset\mathcal{C}\bar{B}[x]$, so $\{O_\lambda\,;\,\lambda\in\Lambda\}$ is an open cover of X with power $\leqq\mathfrak{m}$, and there is a locally finite open refinement $\{Q_\lambda\,;\,\lambda\in\Lambda\}$ with $\overline{Q_\lambda}\subset O_\lambda$ for every $\lambda\in\Lambda$. Let us put

$$G_x=\bigcup_{\overline{Q_\lambda}\ni x}E_\lambda.$$

Take a point a, then there are Q_{λ_0} including a and $U_0\in\mathfrak{N}_a$ such that U_0 meets only a finite number of $\overline{Q_\lambda}$ and $U_0\subset Q_{\lambda_0}$. $\{G_x\,;\,x\in U_0\}$ consists of finitely many different open sets, and if $x\in U_0$, then $x\in Q_{\lambda_0}$ and $G_x\supset E_{\lambda_0}\supset\bar{A}[a]\supset A_a$, so we have

$$c\text{-}\limsup_a G_x\supset\Big(\bigcap_{x\in U_0}G_x\Big)^0=\bigcap_{x\in U_0}G_x\supset A_a.$$

While, there is $U_1\in\mathfrak{N}_a$ such that $U_1\subset U_0$ and $U_1\cap\overline{Q_\lambda}\neq\emptyset$ implies $\overline{Q_\lambda}\ni a$, $O_\lambda\ni a$ and $B_a\cap\overline{E_\lambda}=\emptyset$. Let only $\overline{Q_{\lambda_1}},\cdots,\overline{Q_{\lambda_n}}$ meet U_1, then $x\in U_1$ follows

$$G_x \subset \bigcup_{i=1}^{n} E_{\lambda_i},$$

and we have

$$\limsup_{a} G_x \cap B_a \subset \overline{\bigcup_{x \in U_1} G_x} \cap B_a$$
$$\subset (\bigcup_{i=1}^{n} \overline{E_{\lambda_i}}) \cap B_a = \emptyset.$$

Consequently, $X \times Y$ is normal by Proposition 3 and Theorem in [2].

From Proposition 4 in [3] we have

Corollary 1 (H. Tamano [7]). *Let X be paracompact, and let a normal space Y be upper compact at X, then $X \times Y$ is normal.*

Corollary 2. *Let $\mathfrak{m}$ and $\mathfrak{n}$ be any cardinal numbers. If X is an $\mathfrak{m}$-paracompact normal space every point of which has the character $\mathfrak{n}$ or is an $\mathfrak{m}$-point, and if Y is an $\mathfrak{n}$-compact normal space which has an open base of power $\leqq \mathfrak{m}$ for closed sets, then $X \times Y$ is normal.*

Proof is obtained from Corollary 2 to Proposition 4, Proposition 7 and Proposition 8 in [3].

Corollary 3 (K. Morita [5]). *Let X be an $\mathfrak{m}$-paracompact normal space and Y a compact space with an open base of power $\leqq \mathfrak{m}$. Then $X \times Y$ is normal.*

Corollary 4 (K. Morita [5]). *Let X be a paracompact space every point of which has the character $\leqq \mathfrak{m}$, and let Y be normal and $\mathfrak{m}$-compact, then $X \times Y$ is normal.*

As we have seen above and in [3], there are several cases where compactness in known results can be replaced by upper compactness. Let us show another one below.

Definition. A space Y is called *locally upper compact at a space X* if every point of Y has a neighborhood U such that $\bar{U}$ is upper compact at X.

Theorem 2 ([1]). *Let X be paracompact, and let Y be paracompact and locally upper compact at X, then $X \times Y$ is normal.*

Proof.[1] By proposition 1 in [3] we can construct a locally finite closed cover $\{E_\alpha\}$ of Y every member of which is upper compact at X. $\{X \times E_\alpha\}$ is a locally finite closed cover of $X \times Y$, every member of which is normal by Corollary 1 to Theorem 1, so that $X \times Y$ is normal by Theorem 2 in [6].

Corollary (K. Morita [4]). *Let X be paracompact, and let Y be locally compact and paracompact, then $X \times Y$ is normal.*

The following example shows that, in general, there can not exist any assertion giving necessary condition for the normality of $X \times Y$ on each of X and Y separately, except normality; in other words, it

1) This short proof owes to Prof. K. Morita.

indicates that the necessary and sufficient condition for the normality of $X \times Y$ have, dissimilarly to e.g. the compactness, to be stated in close relations between X and Y. This is the remark emphasized also in [1].

Example. First, we remark that the space of ordinal numbers $W(\omega_{\alpha+1}) - W(\omega_\beta) = \{\lambda ; \omega_\beta \leqq \lambda < \omega_{\alpha+1}\}$ with the order topology is $\aleph_\alpha$-compact, a fortiori $\aleph_\alpha$-paracompact, and not $\aleph_{\alpha+1}$-paracompact, where $-1 \leqq \beta \leqq \alpha$, $\omega_{-1} = 1$ and $\aleph_{-1} = 1$.

Let Y be an arbitrary normal space which is $\mathfrak{n}$-compact, $\mathfrak{n} \geqq 1$ (any space is $\mathfrak{n}$-compact for a finite $\mathfrak{n}$), and let $\mathfrak{m}$ be the power of the family of all open sets in Y, where $\mathfrak{m} \geqq \mathfrak{n}$. Take an $\aleph_{\alpha+1} > \mathfrak{m}$, put $X = W(\omega_{\alpha+1})$, $\alpha \geqq 0$, and topologize X as follows. When the cardinal number corresponding to $\beta \in X$ is greater than $\mathfrak{m}$ or not greater than $\mathfrak{n}$, then the neighborhood system of β is one in the usual order topology; and otherwise, β is isolated. Every point of X then has a neighborhood base of power $\leqq \mathfrak{n}$ or is an $\mathfrak{m}$-point, so Y is upper compact at X by Proposition 8 and Corollary 2 to Proposition 4 in [3]. X is furthermore $\mathfrak{m}$-paracompact, and so $X \times Y$ is normal by Corollary 2 to Theorem 1.

References

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