2. *On a Problem of Vanishing Algebras*

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1. Let G be a locally compact group and let m be a left invariant Haar measure on G. For any measurable subset S of G, define Ls to be the subset of $L^1(G)$ consisting of all functions which vanish locally almost everywhere on the complement of S. When Ls forms a subalgebra of $L^1(G)$, we call it a vanishing algebra. The notion and study of vanishing algebras were initiated by A. B. Simon ([1]).

We see easily

1) If S is a semigroup l.a.e., that is, there exists a semigroup T in G such that $G=T$ locally almost everywhere, then Ls is a vanishing algebra.

2) If S is a group l.a.e., then Ls is a selfadjoint vanishing algebra.

Teng-Sun Liu proved in [2] that the converse of 2) is always true and the converse of 1) is true when G is unimodular and S is σ-compact. The problem whether the converse of 1) is always true or not seems to be unknown. In this short note we shall give an affirmative answer to this problem by applying essentially the existence of a translation invariant lifting which is due to A. I. Tulcea and C. I. Tulcea ([3]).

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2. The result of [3] says that for an arbitrary locally compact group G there exists a mapping θ of measurable sets into measurable sets such that

a) $\theta(A)=A$ l.a.e.;

b) $A=B$ l.a.e. implies $\theta(A)=\theta(B)$;

c) $\theta(A\cap B)=\theta(A)\cap\theta(B)$;

d) $\theta(A)=A$ for any open and closed set A;

e) $\theta(xA)=x\theta(A)$ for every $x\in G$.

They call it *a lower density commuting with left translations of* G.

Using the above mapping essentially we can show our following answer.

Theorem. *If Ls is a vanishing algebra, then S is a semigroup locally almost everywhere.*

Proof. Suppose $D(S)$ is the set of all points $x\in G$ such that $m(S\cap V)>0$ for all neighborhoods V of x and suppose $I(S)$ is the set of all $x\in G$ such that $m(S\setminus V)=0$ for some neighborhood V of x. Then $D(S)$ and $I(S)$ both are semigroups and the following facts are known in [1] and [2].

(1) $S\cap D(S)=S$ l.a.e.,

(2) $I(S)\setminus S=\emptyset$ l.a.e.,

(3) $D(S)I(S)\cup I(S)D(S)\subset I(S)$.

Take a σ-compact and open subgroup H of G and fix representatives $\{x_\lambda\}_{\lambda\in\Lambda}$ of each left coset of H. Let $H=\bigcup_{n=1}^{\infty} Cn$, where Cn is an increasing sequence of compact subsets of H and let $S_\lambda, n=x_\lambda Cn\cap S$ for every $\lambda\in\Lambda$ and $n=1,2,3,\cdots$. We put

$$T_0=\bigcup_{\lambda\in\Lambda}\bigcup_{n=1}^{\infty}(\theta(S_\lambda, n)\cap\theta(S_\lambda^{-1}, n)^{-1}).$$

Then T_0 is a measurable set and we see $S=T_0$ l.a.e. by the properties of θ. Then we define

$$T=(T_0\cap D(S))\cup I(S).$$

We can also see $S=T$ l.a.e. using (1) and (2) above. We are going to show that T is a semigroup, $TT\subset T$. However, since $I(S)$ is a semigroup and we have (3) above, the one thing which we have to show is just that $xy\in T$ for every x and y in $T_0\cap D(S)$. Now suppose $x,y\in T_0\cap D(S)$, then there exist S_λ, n and $S_{\lambda'}, n'$ such that $x\in\theta(S_\lambda, n)\cap\theta(S_\lambda^{-1}, n)^{-1}$ and $y\in\theta(S_{\lambda'}, n')\cap\theta(S_{\lambda'}^{-1}, n')^{-1}$.

Let f and g be characteristic functions of $\theta(S_\lambda, n)$ and $\theta(S_{\lambda'}^{-1}, n')^{-1}$ respectively, then f and g must be elements of Ls. Therefore $f*g=h$ which is the convolution of f and g is an element of Ls. The value of h at the point xy is given by

$$m(\theta(S_\lambda, n)\cap xy\,\theta(S_{\lambda'}^{-1}, n')).$$

And since $x\in\theta(S_\lambda, n)$ and $y\in\theta(S_{\lambda'}^{-1}, n)^{-1}$, we have $x\in\theta(S_\lambda, n)\cap xy\theta(S_{\lambda'}^{-1}, n')$, therefore this implies

$$\theta(S_\lambda, n\cap xyS_{\lambda'}^{-1}, n')=\theta(S_\lambda, n)\cap xy\theta(S_{\lambda'}^{-1}, n')\neq\emptyset.$$

Hence we have

$$h(xy)=m(\theta(S_\lambda, n\cap xyS_{\lambda'}^{-1}, n'))>0.$$

Since h is a continuous function, $h(xy)>0$ implies $xy\in I(S)\subset T$. This completes our proof.

References

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[3] A. Tulcea and C. I. Tulcea: On the existence of a lifting commuting with the left translations of an arbitrary locally compact group. Fifth Berkley Symposium on Mathematical Statistics and Probability, Part II, **1**, 63–97 (1967).