38. *On the Cauchy Problem for a Certain Nonlinear Hyperbolic Partial Differential Equation of the Second Order*

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1. Introduction. The uniqueness theorems for generalized solutions of first order quasilinear hyperbolic equations (or systems) were proved by either Holmgren's method [1], [2], or the method of using the potential function [3]-[5].

The purpose of this note is to extend the uniqueness theorems to certain second order quasilinear hyperbolic equations with two independent variables (Section 2) and with $n\,(\geqslant 2)$ independent variables (Section 3). The proofs of Lemma 1 and Theorem 1 in Section 2 are based on the potential function, and Theorem 2 in Section 3 is obtained by Holmgren's method.

In this note we state the results only. Detailed proof will be published elsewhere.

2. The case of two independent variables.

In $\Omega=\{a\leq x\leq b, 0\leq t\leq T, T>0\}$, we consider the following equation

$$(1)\qquad \partial^2 u(x,t)/\partial t^2=\partial A(x,t,u,\partial u/\partial x)/\partial x+B(x,t,u)$$

with initial conditions

$$(2)\qquad u(x,0)=u_0(x),\qquad \partial u(x,0)/\partial t=v_0(x)$$

where $u_0(x)\in \mathrm{Lip}\,[a,b]$ and $v_0(x)\in \mathrm{L}_\infty[a,b]$. We assume that $A(x,t,u,p)$ is of class C^2 with respect to all arguments and satisfies

$$(3)\qquad \partial A(x,t,u,p)/\partial p>0,\qquad \partial^2 A(x,t,u,p)/\partial p^2>0$$

and that $B(x,t,u)$ is of class C^1 with respect to all arguments.

The definition of the generalized solution $u(x,t)$ of the Cauchy problem (1), (2) is the following: (a) $u(x,t)\in \mathrm{Lip}\,(\Omega)$. (b) $u(x,t)$ satisfies the initial conditions (2) and the integral identity

$$(4)\qquad \oint_\Gamma u_t(x,t)dx+A(x,t,u,u_x)dt-\iint_D B(x,t,u)dxdt=0$$

where Γ is an arbitrary piece-wise smooth closed contour, bounding a domain D and lying in Ω. (c) $u_x(x,t)$ possesses the semi-increasing property with respect to t (in the sence of Douglis), i.e., there is a bounded measurable function $v(x,t)$ defined in Ω such that

$$(5)\qquad u_x(x,t)=v(x,t),\qquad \text{a.e. in } \Omega$$

and that

$$(6)\qquad \frac{v(x,t')-v(x,t)}{t'-t} \geqslant -K(t) \quad \text{for } 0<t<t'\leqslant T$$

where $K(t)$ is a nonnegative and non-increasing function of t on the interval $0<t\leqslant T$.

Introducing the potential function:

$$(7)\qquad U(x,t)=\int_\xi^x u(x',t)dx'+\int_0^t (t-s)[A(\xi,s,u(\xi,s),u_x(\xi,s)) - B'(\xi,s)]ds,$$

where ξ is an arbitrary but fixed number such that $\xi \in [a,b]$ and

$$(8)\qquad B'(\xi,s)=\int_{\xi_0(s)}^{\xi} B(x',s,u(x',s))dx',$$

in which $\xi_0(s)$ is some smooth curve in Ω, we obtain a nonlinear integro-differential equation

$$(9)\qquad \partial^2 U/\partial t^2 = A(x,t,\partial U/\partial x, \partial^2 U/\partial x^2)+\int_{\xi_0(t)}^{x} B(x',t,\partial U(x',t)/\partial x)dx'.$$

Now we consider the Cauchy problem for the equation (9) with initial conditions

$$(10)\qquad U(x,0)=\int_\xi^x u_0(x')dx', \qquad \partial U(x,0)/\partial t=\int_\xi^x v_0(x')dx'.$$

The definition of the generalized solution $U(x,t)$ of (9), (10) is the following: (a) $U(x,t)\in C^1(\Omega)$ and $U_t, U_x \in \mathrm{Lip}(\Omega)$. (b) $U(x,t)$ satisfies the equation (9) almost everywhere with (10). (c) U_{xx} possesses the semi-increasing property with respect to t.

Then if $u(x,t)$ is a generalized solution of (1) with (2), $U(x,t)$ defined by (7) is a generalized solution of (9) with (10). Conversely, if there is a generalized solution $U(x,t)$ of (9) with (10), the function defined by

$$u(x,t)=\partial U(x,t)/\partial x$$

is a generalized solution of (1) with (2).

Let M and t_1 be constants such that

$$(11)\qquad M=\max(A_p(x,t,u,p))^{1/2}$$

$$(12)\qquad t_1=\min(T,(\beta-\alpha)/2M)$$

where maximum is taken for (x,t) in Ω, $|u|\leqslant \max_\Omega |U_x(x,t)|$ and $|p|\leqslant \sup_\Omega |U_{xx}(x,t)|$, and α, β are arbitrary numbers such that $a\leq\alpha<\beta\leq b$.

We shall call a trapezoid $T_0^\tau=\{(x,t);\alpha+Mt\leq x\leq\beta-Mt, 0\leq t<\tau\leq t_1\}$ a trapezoid of determinacy for the generalized solution $U(x,t)$ considered if $\xi_0(t)$ belongs to a rectangle:

$$\alpha+M\tau\leq x\leq \beta-M\tau, \qquad 0\leqslant t<\tau\leqslant t_1.$$

Denoting by I_ρ the intersection $T_0^\tau\cap\{t=\rho\}$, we obtain the following lemma:

Lemma. *Let $U_i(x,t), i=1,2,$ be two generalized solutions of the*

Cauchy problem for the equation (9) *with initial data* (10) *and* $E(t)$ *be the integral*

$$E(t)=\int_{I_t}\left[\frac{1}{f(x,t)}(\partial U_1(x,t)/\partial t-\partial U_2(x,t)/\partial t)^2+(\partial U_1(x,t)/\partial x-\partial U_2(x,t)/\partial x)^2\right]dx, \tag{13}$$

where

$$f(x,t)=\int_0^1 A_p(x,t,\theta\partial U_1/\partial x+(1-\theta)\partial U_2/\partial x,\ \theta\partial^2 U_1/\partial x^2+(1-\theta)\partial^2 U_2/\partial x^2)d\theta. \tag{14}$$

Then, in the common trapezoid of determinacy, there exist appropriate positive constants λ *and* μ *such that the quantity*

$$e^{-\mu t}(k(t))^{-\lambda}E(t)$$

decreases monotonically as t increases in the interval $0<t\leqslant t_1$, *where*

$$k(t)=\exp\left\{-\int_t^{t_1}K(\rho)d\rho\right\}.$$

The constant λ *depends on* $M=\max(A_p(x,t,u,p))^{1/2}$, $\sup|\partial^2 U_i/\partial x\partial t|$, $c_1=\max|A_u(x,t,u,p)|$, $c_2=\max|A_{pt}(x,t,u,p)|$, $c_3=\max|A_{up}(x,t,u,p)|$, $c_4=\max|A_{pp}(x,t,u,p)|$, $c_5=\min|A_p(x,t,u,p)|$, $c_6=\max|B_u(x,t,u)|$ *where maximum and minimum are taken over* (x,t) *in* Ω, $|u|\leqslant\max_\Omega|\partial U_i/\partial x|$ *and over* $|p|\leqslant\sup_\Omega|\partial^2 U_i/\partial x^2|$, $i=1,2$. *The constant* μ *is determined from* M, c_6 *and* $b-a$. *If* $B\equiv 0$, *then we may take* $\mu=0$.

As an immediate consequence of the lemma, we have

Theorem 1. *If* $K(t)$ *is summable in* $(0,T)$, *two generalized solutions of the equation* (1), *which satisfy the same initial conditions, coincide almost everywhere in a common trapezoid of determinacy.*

Remark. We see easily that the similar result as the lemma is valid for the Cauchy problem for the equation of the form:

$$\partial^2 u(x,t)/\partial t^2=A(x,t,\partial u/\partial x_1,\cdots,\partial u/\partial x_n,\Delta u)$$

with initial conditions

$$u(x,0)=u_0(x),\qquad \partial u(x,0)/\partial t=v_0(x)$$

where $x=(x_1,\cdots,x_n)$ and $\Delta u=\partial^2 u/\partial x_1^2+\cdots+\partial^2 u/\partial x_n^2$.

3. The case of n independent variables. Let $S=\{(x,t);t\geqslant 0, x\in R^n\}$, $S_T=\{(x,t);0\leqslant t\leqslant T, x\in R^n\}$ and $\tilde{S}_T=\{(x,t);0\leqslant t<T, x\in R^n\}$. Here T is an arbitrary positive number.

We consider the following second order quasilinear partial differential equation

$$\partial^2 u(x,t)/\partial t^2=\sum_{i=1}^n \partial A_i(x,t,u,\nabla u)/\partial x_i+B(x,t,u,\nabla u) \tag{15}$$

with initial conditions

$$u(x,0)=u_0(x),\qquad \partial u(x,0)/\partial t=v_0(x) \tag{16}$$

where $x=(x_1,\cdots,x_n)\in R^n$, $\nabla u=(\partial u/\partial x_1,\cdots,\partial u/\partial x_n)$, $u_0(x)\in \mathrm{Lip}(R^n)$, and $v_0(x) \in L_\infty(R^n)$. We assume that $A_i(x, t, u, p)$ is of class C^2 with respect to all arguments where $p=(p_1, \cdots, p_n)$ and $A_{ij}(x, t, u, p) = \partial A_i(x, t, u, p)/\partial p_j$ satisfy the following conditions:

1) For all x, t, u and p

$$A_{ij}(x, t, u, p) = A_{ji}(x, t, u, p). \tag{17}$$

2) For all x, t, u, p and all real vectors $\xi=(\xi_1, \cdots, \xi_n)$

$$0<\kappa_1 \sum_{i=1}^{n} \xi_i^2 \leqslant \sum_{i,j=1}^{n} A_{ij}(x, t, u, p)\xi_i \xi_j \tag{18}$$

where κ_1 is a positive constant.

3) For all x, t, u, p and for each k $(k=1, \cdots, n)$

$$\sum_{i,j=1}^{n} \partial^2 A_i(x, t, u, p)/\partial p_j \partial p_k \xi_i \xi_j \geqslant 0. \tag{19}$$

We assume that $B(x, t, u, p)$ is of class C^1 with respect to all arguments.

The definition of the generalized solution u of the Cauchy problem (15), (16) is the following: (a) $u(x, t) \in \mathrm{Lip}\,(S_T)$. (b) $u(x, t)$ satisfies the integral identity

$$\iint_{t\geq 0} [u\cdot\phi_{tt} + \sum_{i=1}^{n} A_i(x, t, u, \nabla u)\phi_{x_i} - B(x, t, u, \nabla u)\cdot\phi]dxdt + \int u_0(x)\phi_t(x, 0)dx - \int v_0(x)\phi(x, 0)dx = 0 \tag{20}$$

for any C^2 test function $\phi(x, t)$ with compact support in $\tilde{S}_T$. (c) its first derivatives $u_{x_i}(x, t)$ $(i=1, \cdots, n)$ possess the semi-increasing property with respect to t, i.e., there exist bounded measurable functions $v_i(x, t)$ $(i=1, \cdots, n)$ defined in S_T such that

$$u_{x_i}(x, t) = v_i(x, t) \qquad \text{a.e. in } S_T \tag{21}$$

and that

$$\frac{v_i(x, t') - v_i(x, t)}{t'-t} \geqslant -K(t) \ \text{ for } 0<t<t'\leqslant T, \tag{22}$$

where $K(t)$ is a nonnegative and non-increasing function of t on the interval $0<t\leqslant T$.

Theorem 2. *If $K(t)$ is summable on $(0, T)$, the generalized solution of the Cauchy problem for the equation* (15) *with the initial conditions* (16) *is unique.*

Outline of proof for Theorem 2. Let $u_1(x, t)$, $u_2(x, t)$ be two generalized solutions of the equation (15) with the same initial data. Then the difference $w(x, t) = u_1(x, t) - u_2(x, t)$ satisfies

$$\iint_{t\geq 0} [w\cdot\phi_{tt} + \sum_{i,j=1}^{n} \tilde{A}_{ij}(x, t)w_{x_j}\phi_{x_i} + \sum_{i=1}^{n} \tilde{A}_{iu}(x, t)w\phi_{x_i} - \sum_{i=1}^{n} \tilde{B}_i(x, t)w_{x_i}\phi - \tilde{B}_u(x, t)w\phi]dxdt = 0 \tag{23}$$

where

$$\tilde{A}_{ij}(x, t) = \int_0^1 A_{ij}(x, t, \theta u_1 + (1-\theta)u_2, \theta\nabla u_1 + (1-\theta)\nabla u_2)d\theta,$$

$$\tilde{A}_{iu}(x,t)=\int_0^1 \partial A_i(x,t,\theta u_1+(1-\theta)u_2,\theta\nabla u_1+(1-\theta)\nabla u_2)/\partial u d\theta,$$

$$\tilde{B}_i(x,t)=\int_0^1 \partial B(x,t,\theta u_1+(1-\theta)u_2,\theta\nabla u_1+(1-\theta)\nabla u_2)/\partial p_i d\theta,$$

$$\tilde{B}_u(x,t)=\int_0^1 \partial B(x,t,\theta u_1+(1-\theta)u_2,\theta\nabla u_1+(1-\theta)\nabla u_2)/\partial u d\theta.$$

We shall establish that $w=0$ by showing

$$\iint_{t\geq 0} \Phi(x,t)w(x,t)dxdt=0 \tag{24}$$

for any C^2-function Φ with compact support in $\tilde{S}_T$.

By assumptions, there exist positive constants $\kappa_1, \kappa_2, c_1, c_2, c_3$ and a function $L(t)$ such that

$$0<\kappa_1\sum_{i=1}^n \xi_i^2 \leq \sum_{i,j=1}^n \tilde{A}_{ij}(x,t)\xi_i\xi_j \leq \kappa_2 \sum_{i=1}^n \xi_i^2,$$

$$|\tilde{A}_{iu}|\leqslant c_1,\ |\tilde{B}_i w_{x_i}|\leqslant c_2,\ |\tilde{B}_u|\leqslant c_3,\ \sum_{i,j=1}^n \frac{\tilde{A}_{ij}(x,t')-\tilde{A}_{ij}(x,t)}{t'-t}\xi_i\xi_j \geqslant$$

$$-L(t)\sum_{i=1}^n \xi_i^2,\ 0<t<t'\leqslant T$$

for all real vectors ξ and for any bounded domain in S_T. Here $L(t)$ is nonnegative and non-increasing on the interval $0<t\leqslant T$ (note that, if $K(t)$ is summable on $(0,T)$, $L(t)$ is also summable on it). Then by a familiar argument we may construct sequences of functions $\{A_{i,j}^m(x,t)\}$, $\{A_{iu}^m(x,t)\}$, $\{B_i^m(x,t)\}$, $\{B_u^m(x,t)\}$ which are infinitely differentiable and converge in $L^2_{\mathrm{loc}}(S_T)$ as $m\to\infty$ to $\tilde{A}_{ij}(x,t)$, $\tilde{A}_{iu}(x,t)$, $\tilde{B}_i(x,t)w_{x_i}$, $\tilde{B}_u(x,t)$, respectively and satisfy

$$\begin{gathered}0<\kappa_1\sum_{i=1}^n \xi_i^2 \leq \sum_{i,j=1}^n A_{ij}^m(x,t)\xi_i\xi_j \leq \kappa_2\sum_{i=1}^n \xi_i^2,\\ |A_{ij}^m(x,t)|\leqslant c_1,\ |B_i^m(x,t)|\leqslant c_2,\ |B_u^m(x,t)|\leqslant c_3,\\ \sum_{i,j=1}^n \partial A_{ij}^m(x,t)/\partial t\xi_i\xi_j \geqslant -L(t)\sum_{i=1}^n \xi_i^2\end{gathered} \tag{25}$$

for all real vectors ξ and for any bounded domain in S_T.

We now consider the backward Cauchy problem of the equation

$$\begin{aligned}\partial^2\phi^m/\partial t^2 &= \sum_{i,j=1}^n \partial(A_{ij}^m(x,t)\partial\phi^m/\partial x_i)/\partial x_j - \sum_{i=1}^n A_{iu}^m(x,t)\partial\phi^m/\partial x_i\\ &\quad -\sum_{i=1}^n B_i^m(x,t)\phi^m - B_u^m(x,t)\phi^m = \Phi(x,t)\end{aligned} \tag{26}$$

with initial conditions

$$\phi^m(x,T)=\partial\phi^m(x,T)/\partial t=0 \tag{27}$$

In virtue of the conditions (24) and the summability of $L(t)$, it is easily to show the fact that $\partial\phi^m/\partial x_i, \phi^m$ are uniformly bounded in $L^2_{\mathrm{loc}}(S_T)$, from which the validity of the relation (24), i.e., the conclusion of Theorem 2, immediately follows.

Remark. If we are concerned with the equation (15) with conditions (19) replaced by

(19′) $$\sum_{i,j=1}^{n} \partial^2 A_i(x, t, u, p)/\partial p_j \partial p_k \xi_i \xi_j \leqslant 0,$$

we must replace the inequality (22) by

(22′) $$\frac{v_i(x, t') - v_i(x, t)}{t' - t} \leqslant J(t) \text{ for } 0 < t < t' \leqslant T,$$

where $J(t)$ is a nonpositive and non-decreasing function of t on the interval $0 < t \leqslant T$.

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