37. On a Riemann Definition of the Stochastic Integral. II

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§ 3. Riemann definition of the stochastic integral.

Now let us get back to the original problem stated in the first paragraph.

As we shall see soon later, the ratio k of interpolation in the Riemann sum (1) performs an important role in this problem, henceforth we shall rewrite the sum $S_n(f)$ into $I_k^{(n)}(f)$ so as to emphasize this ratio k.

Let $\mathcal{S}$ be the class of functions $f_t(\omega)$ which satisfy the conditions $(s,1)$, $(s,2)$ and the following

$(S,3)$ $f_t(\omega)$ is β^+-differentiable in the L_4-sense on $[0,T]$ and its β^+-derivative is Riemann integrable on $[0,T]$ in the L_2-sense.

Now as for the sequence $\{I_k^{(n)}(f)\}(n=1,2,\cdots)$, next Theorem 5 assures the existence of its limit.

Theorem 5. *Let $\{\Delta^{(n)}\}$ be a suquence of canonical partitions on $[0,T]$. Then for an arbitrary $f_t(\omega)$ $(\in\mathcal{S})$ and an arbitrary number k $(0\leqslant k\leqslant 1)$, the limit of the sequence exists and is given by*

$$\text{(10)}\qquad \underset{n\to\infty}{\text{l.i.m}}\, I_k^{(n)}(f)(\omega)\equiv I_k(f)(\omega)=I_0(f)(\omega)+k\cdot\int_0^T\frac{\partial^+}{\partial^+\beta_t}f_t(\omega)dt.$$

Definition. We shall call this limit $I_k(f)(\omega)$ the stochastic integral of $f_t(\omega)$ of index k.

As mentioned in § 1, thus constructed stochastic integral $I_k(f)(\omega)$ $(k\neq 0)$ is a generalization of the stochastic integral introduced by R. L. Stratonovich.

For the class of functions which can be represented in the form $\varphi(t,\xi_t(\omega))$ stated in the Example 3, R. L. Stratonovich has defined his integral as the limit in the mean of the following series (see R. L. Stratonovich (2)).

$$\text{(11)}\qquad \int_0^T\varphi(t,\xi_t(\omega))d^*\beta_t(\omega)$$
$$\equiv\underset{n\to\infty}{\text{l.i.m}}\sum_i\varphi(t_i^{(n)},k\xi_{t_{i+1}^{(n)}}+(1-k)\xi_{t_i^{(n)}})(\beta_{t_{i+1}^{(n)}}(\omega)-\beta_{t_i^{(n)}})(\omega)$$
$$t_i^{(n)}\in\Delta^{(n)}\quad\text{and}\quad 0<k\leqslant 1$$

*) Rigorously speaking, Stratonovich has defined his integral $\int_0^T\varphi(t,\xi_t(\omega))d\xi_t(\omega)$ only in case of $k=1/2$, in the above definition (11), but the essence of the way of the definition is not much different.

where the notation $d^*\beta_t$ means the integral in the Stratonovich sense.

Under additional conditions (see the conditions (2) in his article (2)), he obtained the following results, that the integral defined by (11) exists and that its limit satisfies the following relation,

$$(12)\quad \int_0^T \varphi(t, \xi_t(\omega))d^*\beta_t(\omega) = \int_0^T \varphi(t, \xi_t(\omega))d^0\beta_t(\omega) + k\cdot\int_0^T b(t, \xi_t(\omega))\frac{\partial}{\partial x}\varphi(t,\xi_t(\omega))dt$$

where $b(t, \xi_t(\omega))$ is the diffusion coefficient at t of the diffusion process $\xi_t(\omega)$.

However it is easily be seen by Example 3 in § 2 and (12) that for the functions which are integrable in the sense of Stratonovich are also integrable in our sense, and its integral $I_k(f)(\omega)$ coincide with the integral in the sense of Stratonovich. Moreover we can see that the class of functions which are integrable in our sense is much wider than that of Stratonovich, because the processes which are β-differentiable in the L_4-sense are not limitted in classical diffusion processes.

Thus, the integral $I_k(f)(\omega)$ can be regarded as a generalized form of the stochastic integral introduced by Stratonovich.

§ 4. Stochastic integral $I_{1/2}(f)(\omega)$ as a limit of a sequence of Stieltjes integrals.

Let $\{\varDelta^{(n)}\}$ be a sequence of partitions of the interval $[0, T]$ such that,

$$\varDelta^{(n)};\ 0=t_0^{(n)}<t_1^{(n)}<\cdots<t_{2^n}^{(n)}=T, \quad \text{where} \quad t_i^{(n)}=\frac{i}{2^n}T,$$

then we have $\varDelta^{(n)}\subset\varDelta^{(n+1)}$

Let us make linear approximation process $\beta_t^{(n)}(\omega)$ of the Brownien motion process $\beta_t(\omega)$ in the following manner

$$(13)\quad \beta_t^{(n)}(\omega)=\beta_{t_i^{(n)}}(\omega)+(\beta_{t_{i+1}^{(n)}}(\omega)-\beta_{t_i^{(n)}}(\omega))\frac{t-t_i^{(n)}}{\tau^{(n)}} \quad \text{for} \quad t\in[t_i^n, t_{i+1}^n),$$

where $\tau^{(n)}=T/2^n$.

It will easily be seen that the stochastic crocess $\beta_t^{(n)}(\omega)$ converges in the mean uniformly to $\beta_t(\omega)$ as n tends to infinity.

Now our next aim is to find conditions under which the following sequence of Stieltjes integrals converges.

$$(14)\quad S(f_t^{(n)}(\omega))=\int_0^T f_t^{(n)}(\omega)d\beta_t^{(n)}(\omega)=\sum_i\frac{\varDelta_i^{(n)}\beta}{\tau^{(n)}}\cdot\int_{t_i^{(n)}}^{t_{i+1}^{(n)}} f_t^{(n)}(\omega)dt$$

where $\varDelta_i^{(n)}\beta=\beta_{t_{i+1}^{(n)}}(\omega)-\beta_{t_i^{(n)}}(\omega)$ and $f_t^{(n)}(\omega)$ are functions which satisfy the following conditions;

f, 1) $f_t^{(n)}(\omega)$ is a $B_{[0,t]}\times N_{[t]}^0$ -measurable function for each $n(=1,2\cdots)$, where $[t]=t_{i+1}^{(n)}$ for $t_i^{(n)}<t\leqslant t_{i+1}^{(n)}$.

f, 2) $\{f_t^{(n)}(\omega)\}$ converges in the L_2-sense to a function $f_t(\omega)$ in $\mathcal{S}$ for each t as n tends to infinity.

We can show that the sequence converges under similar conditions as in § 3.

To explain, this, let us begin with the following definition.

Definition. Let $\{f_t^{(n)}(\omega)\}$ be a family of functions satisfying the conditions (f, 1) and (f, 2). If there exists a sequence of random vectors $\{\zeta^{(n)}_{t_1^{(n)}}, \zeta^{(n)}_{t_2^{(n)}}, \cdots \zeta^{(n)}_{t_{2^n}^{(n)}}\}$ $(n=1, 2, \cdots)$ component $\zeta_{t_i^{(n)}}^{(n)}(\omega)$ of which satisfies the following conditions,

$\zeta^{(n)}, 1)$ $\zeta_p^{(n)}(\omega)$ is N_p^0 -measurable,

$\zeta^{(n)}, 2)$ $M|\zeta_p^{(n)}(\omega)|^{2m} < +\infty$

$\zeta^{(n)}, 3)$

$$\lim_{n\to\infty} M\left\{\frac{1}{\sqrt{\tau^{(n)}}}[f^{(n)}_{p+\tau(n)}(\omega)-f_p^{(n)}(\omega)-\zeta_p^{(n)}(\omega)(\beta^{(n)}_{p+\tau(n)}(\omega)-\beta_p^{(n)}(\omega)]\right\}^{2m}=0$$

then we that the family $\{f_t^{(n)}(\omega)\}$ is $\beta^{(n)}$-differentiable in the L_{2m}-sense and call the sequence of vectors $\{\zeta_p^{(n)}\}$ its $\beta^{(n)}$-derivatives with respect to the partition $\Delta^{(n)}$.

Of course, such a sequence of vectors $\{\zeta_p^{(n)}\}$ is not necessarily unique, however, the limit of the sequence, if it exists, is unique as we can see easily.

With this preparation, we can give an answer to the problem in following.

Theorem 6. *We assume that a family $\{f_t^{(n)}(\omega)\}$ satisfies the conditions* f, 1) *and* f, 2) *and that $f_t^{(n)}(\omega)$ is $\beta^{(n)}$-differentiable in the L_4-sense with respect to the partitions $\{\Delta^{(n)}\}$ just defined at the top of this paragraph, and that each of the random variables $\zeta_p^{(n)}(\omega)$ $(n=0, 1, \cdots;$ $p \in \Delta^{(n)})$ satisfies the following*

f, 3) $$\lim_{n\to\infty} M|\zeta_p^{(n)}(\omega)-\zeta_p(\omega)|^2=0 \text{ for } p=\frac{i}{2^m}\cdot T$$

$$\text{where} \quad \zeta_p(\omega)=\frac{\partial^+}{\partial^+\beta_t}f_t(\omega)|_{t=p} \quad \text{and} \quad f_t(\omega)$$

is the function described in the condition f, 2).

Then we have

$$\text{(15)} \quad \underset{n\to\infty}{\text{l.i.m}}\, S(f_t^{(n)}(\omega))=I_0(f)(\omega)+\frac{1}{2}\int_0^T \frac{\partial^+}{\partial^+\beta_t}f_t(\omega)dt \equiv I_{\frac{1}{2}}(f)(\omega)$$

The proof of Theorem 6 will be completed with the help of the next lemma.

Lemma. *For the family of functions $\{f_t^{(n)}(\omega)\}$ and the function just mentioned above, the following relation holds*

$$\text{(16)} \quad \underset{n\to\infty}{\text{l.i.m}} \sum_i \frac{\Delta_i^{(n)}\beta}{\tau^{(n)}}\zeta^{(n)}_{t_i^{(n)}}(\omega)\int_{t_i^{(n)}}^{t_{i+1}^{(n)}}(\beta_t^{(n)}-\beta^{(n)}_{t_i^{(n)}})dt=\frac{1}{2}\int_0^T\frac{\partial^+}{\partial^+\beta_t}f_t(\omega)dt.$$

Theorem 6 is an expected result from the work of Wong and Zakai (Wong and Zakai (4)).

Example. Let $\varphi(x)$ be a function with the bounded and Lipshitzcontinuous first derivative, and let $\beta_t^{(n)}(\omega)$ be the stochastic process defined by (13).

Then the functions $\varphi(\beta_t(\omega))$ and $\varphi(\beta_t^{(n)}(\omega))(n=1,2,\cdots)$ satisfy f, 1) $\sim$f, 3), so the formula (15) holds in this case.

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