30. On Vector Measures. II

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In [4] we have proved the following theorem. *Let S be a set, R a semi-tribe (δ-ring) of subsets of S, X a normed space and m; $R \to X$ a vector measure. Then there exists a finite non-negative measure ν on R such that*

(1) *for any $A \in R$ and any number $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon, A) > 0$ such that $B \in R$, $B \subset A$ and $\nu(B) < \delta \Rightarrow \|m(B)\| < \varepsilon$*

(2) $\nu(E) \leqq \sup \{\|m(A)\| ; A \subset E, A \in R\}$ *for $E \in R$* ([4] *Theorem* 1).

The purpose of this paper is to point out some properties of regular vector measures by using this theorem. These properties were proved earlier (Dinculeanu [1] § 16, Theorem 3, Corollaries 1–4) for vector measures with finite variation, but we shall drop this condition and we shall consider the necessary and sufficient condition for the extension of a regular, finitely additive set function from some clan to a wider class of subsets (cf. Theorem 3). And Corollary 1 is the extension of Dinculeanu's and Kluvanek's result ([2] Theorem 5).

3. Regular vector measures. Suppose that S be a locally compact, Hausdorff space and X a Banach space.

Definition 3. Let R be a clan (ring) of subsets of S. A set function m; $R \to X$ is called regular if for every $A \in R$ and every number $\varepsilon > 0$ there exists a compact set $K \subset A$ and an open set $G \supset A$ such that for every $A' \in R$ with $K \subset A' \subset G$ we have $\|m(A) - m(A')\| < \varepsilon$.

Definition4. Let m; $R \to X$ be a set function and μ a non-negative measure on R. m is μ-absolutely continuous if for every $A \in R$ and every number $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon, A) > 0$ such that for every $B \in R$ with $B \subset A$ and $\mu(B) < \delta$ we have $\|m(B)\| < \varepsilon$.

Lemma 2. *Let R be semi-tribe of subsets of S which has the following conditions*

(*) *for every compact set K and for every open set G such that $K \subset G$, there exists a $A \in R$ such that $K \subset A \subset G$.*

If m; $R \to X$ is a regular vector measure, then there exists a finite non-negative measure ν on R such that

(1) *m is ν-absolutely continuous.*

(2) $\nu(E) \leqq \sup \{\|m(A)\| ; A \subset E, A \in R\}$ *for $E \in R$.*

(3) *ν is regular.*

Proof. It is easy by [4] Theorem 1.

Theorem 3. *Let R be a clan which has the following conditions*

(∗) *for every compact set K and for every open set G such that* $G\supset K$ *there exists a* $A\in R$ *such that* $K\subset A\subset G$.

(∗∗) *for every* $A\in \boldsymbol{R}$ *there exists a* $A'\in R$ *such that* $A\subset \operatorname{Int}A'$.

Then every regular and finitely additive set function m ; $R\to X$ *can be extended uniquely to a regular vector measure* m_1, *on the semi-tribe* φ *generated by* R *if and only if there exists a finite, non-negative, regular measure* ν *on* R *such that* m *is* ν*-absolutely continuous. In this case,* m *becomes countably additive.*

Proof. The necessity is immediate by Lemma 2.

Sufficiency. By Dinculeanu ([1] § 16, Theorem 2, Corollary 2) ν can be extended uniquely to a finite, non-negative regular measure ν_1 on φ. For any $A\in\varphi$ there exists a $E\in R$ with $E\supset A$ (Dinculeana [1] § 1, Proposition 10, corollary). By ν-absolute continuity of m, for every $\varepsilon>0$ there exists a number $\delta=\delta(\varepsilon\cdot A)>0$ such that $B\in R$, $B\subset E$ and $\nu_1(B)=\nu(B)<\delta\Rightarrow\|m(B)\|<\varepsilon$. Hence B, $C\in R$, $B\subset E$, $C\subset E$ and $\nu_1(B\varDelta C)=\nu(B\varDelta C)<\delta\Rightarrow$

$$\begin{aligned}\|m(B)-m(C)\|&=\|m(B-C)-m(C-B)\|\\&\leqq\|m(B-C)\|+\|m(C-B)\|<2\varepsilon\end{aligned}$$

Since ν_1 is regular, for above A and δ there exists a compact set $K\subset A$ and an open set $G\supset A$ such that $B\in R$ and $B\subset G-K$ implies $\nu_1(B)<\delta$.

By (∗) there exists a $B\in R$ with $K\subset B\subset G$. Therefore $B\cap E\in R$ and $A\varDelta(B\cap E)\subset G-K$ implies $\nu_1(A\varDelta(B\cap E))<\delta$. Now we take $B_1\in R$ and $B_2\in R$ such that $\nu_1(A\varDelta(B_1\cap E))<\frac{1}{2}\delta_0$ and

$$\nu_1(A\varDelta(B_2\cap E))<\frac{1}{2}\delta_0 \quad \text{for any} \quad \delta_0(0<\delta_0\leqq\delta).$$

Then

$$\begin{aligned}\nu((B_1\cap E)\varDelta(B_2\cap E))&=\nu_1((B_1\cap E)\varDelta(B_2\cap E))\\&\leqq\nu_1(A\varDelta(B_1\cap E))+\nu_1(A\varDelta(B_2\cap E))<\delta_0\leqq\delta.\end{aligned}$$

Hence we have $\|m(B_1\cap E)-m(B_2\cap E)\|<2\varepsilon$. Since X is complete space, we have $m_E(A)=\lim_{\nu_1(A\varDelta(B\cap E))\to 0} m(B\cap E)$. In particular, $m_E(A)=m(A)$ for $A\in R$. The uniqueness of m_E is clear.

Next we shall prove that m_E is independent on $E(\supset A)$. For any $F\in R$ with $A\subset F\subset E$ and every number $\varepsilon>0$, there exists $\delta_1=\delta(\varepsilon,E)$ and $\delta_2=\delta(\varepsilon,F)>0$. We set $\delta=\min(\delta_1,\delta_2)$. We take B_1, $B_2\in R$ such that $\nu_1(A\varDelta(B_1\cap E))<\frac{1}{2}\delta$ and $\nu_1(A\varDelta(B_2\cap E))<\frac{1}{2}\delta$. Then $\nu_1((B_1\cap E)\varDelta(B_2\cap F))<\delta\leqq\delta_1$ and $B_2\cap F\subset E$.

It follows that $\|m(B_1\cap E)-m(B_2\cap F)\|<2\varepsilon$. Therefore $\|m_E(A)-m_F(A)\|\leqq 2\varepsilon$. Since ε is arbitrary, we have $m_E(A)=m_F(A)$. For every $F\in R$ with $F\supset A$, $m_F(A)=m_{E\cap F}(A)=m_E(A)$.

If we put $m_1(A)=m_E(A)$ $(A\subset E\in R)$, we have

(i) m_1 is finitely additive.

(ii) m_1 is ν_1-absolutely continuous.

(iii) m_1 is regular.

(iv) m_1 is countably additive.

Since (i) is clear (see Kluvanek [3] Theorem 1), (iii) is clear by (**) and (ii), and (iv) is clear by (ii), it only remains to prove (ii): For any $E \in \varphi$ there exists a $F \in R$ with $F \supset E$. From ν-absolute continuity of m for every number $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon, F) > 0$ such that $A \subset F$ $A \in R$ and $\nu(A) < \delta$ implies $\|m(A)\| < \frac{1}{2}\varepsilon$. Let $B \in \varphi$ be a set such that $B \subset E$ and $\nu_1(B) < \delta$. We put $\delta_1 = \nu_1(B)$. Then from the definition of $m_1(B)$, there exists a number $\delta_2 = \delta(\varepsilon, E) > 0$ such that $\nu_1(B \Delta (B_1 \cap F)) < \delta_2$ implies $\|m_1(B) - m(B_1 \cap F)\| < \frac{1}{2}\varepsilon$. We put $\delta_0 = \min(\delta - \delta_1, \delta_2)$. Let $B_1 \in R$ be a set with $\nu_1(B \Delta (B_1 \cap F)) < \delta_0$. Then

$$\|m_1(B) - m(B_1 \cap F)\| < \frac{1}{2}\varepsilon.$$

$$\begin{aligned} |\nu_1(B) - \nu(B_1 \cap F)| &= |\nu_1(B - B_1 \cap F) - \nu_1(B_1 \cap F - B)| \\ &\leqq \nu_1(B \Delta (B_1 \cap F)) < \delta_0 \leqq \delta - \delta_1 \end{aligned}$$

so $\nu(B_1 \cap F) < \nu_1(B) + \delta - \delta_1 = \delta$. Therefore $\|m(B_1 \cap F)\| < \frac{1}{2}\varepsilon$. Thus we have $\|m_1(B)\| \leqq \|m_1(B) - m(B_1 \cap F)\| + \|m(B_1 \cap F)\| < \varepsilon$. The uniqueness of m_1 is clear by the uniqueness of m_E. Q.E.D.

Denote by $\mathfrak{B}_0$ the semi-tribe of the relatively compact Baire sets, by $\mathfrak{B}$ the semi-tribe of the relatively compact Borel sets and by $\mathfrak{R}_0$ the clan generated by the compact sets with are G_δ.

Corollary 1. *Let R be a clan such that $\mathfrak{R}_0 \subset R \subset \mathfrak{B}$. every regular and finitely additive set function $m: R \to X$ can be extended uniquely to a regular Borel measure m; $\mathfrak{B} \to X$ if and only if there exists a finite, non-negative, regular measure ν on **R** such that m is ν-absolutely continuous. In this case m becomes countably additive.*

Proof. By Dinculeanu ([1] §14, Propositions 11 and §15, Lemma 1) R is satisfied the conditions (*), (**) of Theorem 3. Then we can prove in the same way as the proof of Theorem 3.

If we put $R = \mathfrak{B}_0$ Then we have the following result.

Corollary 2. *Every Baire measure m; $\mathfrak{B}_0 \to X$ can be extended uniquely to a regular Borel measure m_1; $\mathfrak{B} \to X$* (Dinculeanu and Kluvanek [2] Theorem 5).

References

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