27. *Axioms for Commutative Rings*

By Saburo TAMURA

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G. R. Blakley, S. Ôhashi and K. Iséki gave some new definitions of commutative rings and semirings (see [1]-[3]). In this note, we shall give other difinitions of commutative rings with unity and semirings with zero and unity, where two binary operations are commutative.

Theorem 1. *A set with two nullary operations,* 0 *and* 1, *with one unary operation,* $-$, *and with two binary operations,* $+$ *and juxtaposition, such that*

1.1) $r+0=r$,

1.2) $r1=r$,

1.3) $((-r)+r)a=0$,

1.4) $(a+(b+cz))r=(br+ar)+z(cr)$

for any a, b, c, r, z, *is a commutative ring with unity.*

Proof. We can prove this theorem as follows.

1.5) $(-r)+r=0$ (See [1])

1.6) $0a=0$ (See [1])

1.7) $a+b$

$=(a+(b+00))1$ by 1.6, 1.1, 1.2.

$=(b1+a1)+0(01)$ by 1.4.

$=b+a$ by 1.2, 1.6, 1.1.

1.8) cz

$=(0+(0+cz))1$ by 1.7, 1.1, 1.2.

$=(01+01)+z(c1)$ by 1.4.

$=zc$ by 1.7, 1.2, 1.1.

1.9) $a+(b+c)$

$=(a+(b+c1))1$ by 1.2.

$=(b1+a1)+1(c1)$ by 1.4.

$=(a+b)+c$ by 1.7, 1.8, 1.2.

1.10) $(zc)r$

$=(0+(0+cz))r$ by 1.8, 1.7, 1.1.

$=(0r+0r)+z(cr)$ by 1.4.

$=z(cr)$ by 1.6, 1.7, 1.1.

1.11) $(b+c)r$

$=(0+(b+c1))r$ by 1.7, 1.1, 1.2.

$=(br+0r)+1(cr)$ by 1.4.

$=br+cr$ by 1.6, 1.1, 1.8, 1.2.

1.12) For given a, b, $a+x=b$ is solvable. (See [2])

Thus this set is a commutative ring with unity. Therefore the proof of Theorem 1 is complete.

Theorem 2. *A set with two nullary operations,* 0 *and* 1, *with one unary operation,* $-$, *and with two binary operations,* $+$ *and juxtaposition, such that*

2.1) $r+0=0+r=r$,

2.2) $((-r)+r)a=0$,

2.3) $(a+(b+cz))r+s=((br+ar)+z(cr))+s1$

for any a, b, c, r, s, z, *is a commutative ring with unity.*

Proof. We can prove this theorem as follows.

2.4) $(-0)a$
$=((-0)+0)a$ by 2.1.
$=0$ by 2.2.

2.5) $0r+s1$
$=((0r+(-0)r)+(-0)((-0)r))+s1$ by 2.4, 2.1.
$=((-0)+(0+(-0)(-0)))r+s$ by 2.3.
$=s$ by 2.4, 2.1.

2.6) $0r$
$=0r+(-0)1$ by 2.4, 2.1.
$=-0$ by 2.5.

2.7) $(-0)+(-0)$
$=0r+01$ by 2.6.
$=0$ by 2.5.

2.8) $s1$
$=(((-0)+(-0))+0)+s1$ by 2.7, 2.1.
$=((0r+0r)+(-0)(0r))+s1$ by 2.6, 2.4.
$=(0+(0+0(-0)))r+s$ by 2.3.
$=(-0)r+s$ by 2.6, 2.1.
$=s$ by 2.4, 2.1.

2.9) $(a+(b+cz))r$
$=(a+(b+cz))r+0$ by 2.1.
$=((br+ar)+z(cr))+01$ by 2.3.
$=(br+ar)+z(cr)$ by 2.8, 2.1.

The remaining part of the proof can be trivially given by using Theorem 1. Therefore the proof of Theorem 2 is complete.

Theorem 3. *A set with two nullary operations,* 0 *and* 1, *with one unary operation,* $-$, *and with two binary operations,* $+$ *and juxtaposition, such that*

3.1) $r+0=0+r=r$,

3.2) $r1=r$,

3.3) $(a+(b+cz))r+((-t)+t)d=(br+ar)+z(cr)$

for any a, b, c, d, r, t, z, *is a commutative ring with unity.*

Proof. We can prove this theorem as follows.

3.4) $(-0)d$

$=(0+(0+01))1+((-0)+0)d$ by 3.2, 3.1.

$=(01+01)+1(01)$ by 3.3.

$=10$ by 3.2, 3.1.

3.5) 10

$=(-0)1$ by 3.4.

$=-0$ by 3.2.

3.6) $(-0)d$

$=-0$ by 3.4, 3.5.

3.7) $(-0)+(-0)$

$=(0+(0+10))1+((-0)+0)1$ by 3.5, 3.1, 3.2.

$=(01+01)+0(11)$ by 3.3.

$=0$ by 3.2, 3.1.

3.8) -0

$=(01+01)+(-0)((-0)1)$ by 3.6, 3.2, 3.1.

$=(0+(0+(-0)(-0)))1+((-0)+0)1$ by 3.3.

$=(-0)+(-0)$ by 3.6, 3,1, 3.2.

$=0$ by 3.7.

3.9) $((-t)+t)d$

$=(0+(0+01))1+((-t)+t)d$ by 3.2, 3.1.

$=(01+01)+1(01)$ by 3.3.

$=0$ by 3.2, 3.1, 3.5, 3.8.

3.10) $(a+(b+cz))r$

$=(br+ar)+z(cr)$ by 3.3, 3.9, 3.1.

The remaining part of the proof can be trivially given by using Theorem 1.

Theorem 4. *A set with two nullary operations,* 0 *and* 1, *with one unary operation,* $-$, *and with two binary operations,* $+$ *and juxtaposition, such that*

4.1) $r+0=0+r=r,$

4.2) $01=10=0,$

4.3) $(a+(b+cz))r+(s+((-t)+t)d)=((br+ar)+z(cr))+s1$

for any a, b, c, d, r, s, t, z, *is a commutative ring with unity.*

Proof. We can prove this theorem as follows.

4.4) $((-t)+t)d$

$=(0+(0+01))1+(0+((-t)+t)d)$ by 4.2, 4.1.

$=((01+01)+1(01))+01$ by 4.3.

$=0$ by 4.2, 4.1.

4.5) $(a+(b+cz))r+s$

$=((br+ar)+z(cr))+s1$ by 4.3, 4.4, 4.1.

The remaining part of the proof can be trivially given by using Theorem 2.

Theorem 5. *A set with two nullary operations,* 0 *and* 1, *and with two binary operations,* + *and juxtaposition, such that*

5.1) $r+0=r$,

5.2) $r1=r$,

5.3) $0a=0$,

5.4) $(a+(b+cz))r=(br+ar)+z(cr)$

for any a, b, c, r, z, *is a semiring with* 0 *and* 1, *where these binary operations satisfy the commutative laws.*

Proof. We can prove this theorem by the same method as Theorem 1.

Theorem 6. *A set with two nullary operations,* 0 *and* 1, *and with two binary operations,* + *and juxtaposition, such that*

6.1) $r+0=0+r=r$,

6.2) $0a=0$,

6.3) $(a+(b+cz))r+s=((br+ar)+z(cr))+s1$

for any a, b, c, r, s, z, *is a semiring with* 0 *and* 1, *where these binary operations satisfy the commutative laws.*

Proof. We can prove this theorem as follows.

6.4) s

$=(0+(0+00))r+s$	by 6.2, 6.1.
$=((0r+0r)+0(0r))+s1$	by 6.3.
$=s1$	by 6.2, 6.1.

6.5) $(a+(b+cz))r$

$=(a+(b+cz))r+0$	by 6.1.
$=((br+ar)+z(cr))+01$	by 6.3.
$=(br+ar)+z(cr)$	by 6.2, 6.1.

The remaining part of the proof can be trivially given by using Theorem 5.

Theorem 7. *A set with two nullary operations,* 0 *and* 1, *and with two binary operations,* + *and juxtaposition, such that*

7.1) $r+0=0+r=r$,

7.2) $r1=r$,

7.3) $(a+(b+cz))r+0d=(br+ar)+z(cr)$

for any a, b, c, d, r, z, *is a semiring with* 0 *and* 1, *where these binary operations satisfy the commutative laws.*

Proof. We can prove this theorem as follows.

7.4) $0d$

$=(0+(0+01))1+0d$	by 7.2, 7.1.
$=(01+01)+1(01)$	by 7.3.
$=10$	by 7.2, 7.1.

7.5) 10

$=01$	by 7.4.
$=0$	by 7.2.

7.6) $(a+(b+cz))r$
$=(br+ar)+z(cr)$ by 7.3, 7.4, 7.5, 7.1.

The remaining part of the proof can be trivially given by using Theorem 5.

Theorem 8. *A set with two nullary operations,* 0 *and* 1, *and with two binary operations,* + *and juxtaposition, such that*

8.1) $r+0=0+r=r$,

8.2) $01=0$,

8.3) $(a+(b+cz))r+(s+0d)=((br+ar)+z(cr))+s1$

for any a, b, c, d, r, s, z, *is a semiring with* 0 *and* 1, *where these binary operations satisfy the commutative laws.*

Proof. We can prove this theorem as follows.

8.4) $0d=10$ (See 7.4)

8.5) $10=0$ (See 7.5)

8.6) $(a+(b+cz))r+s=((br+ar)+z(cr))+s1$ (See 7.6)

The remaining part of the proof can be trivially given by using Theorem 6.

References

[1] G. R. Blakley: Four axioms for commutative rings. Notices of Amer. Math. Soc., **15**, p. 730 (1968).

[2] S. Ôhashi: On axiom systems of commutative rings. Proc. Japan Acad., **44**, 915–919 (1968).

[3] K. Iséki and S. Ôhashi: On definitions of commutative rings. Proc. Japan Acad., **44**, 920–922 (1968).