21. *Endomorphism Rings of Modules over Orders in Artinian Rings*

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Recently Small [7] proved that if a ring R has a right Artinian (classical) right quotient ring, then so does the endomorphism ring of a finitely generated projective right R-module.

On the other hand, it has been shown by Hart [3] that if a ring R has a semi-simple Artinian (classical) two-sided quotient ring Q, so does the endomorphism ring of a finitely generated torsion free right R-module M. In this case M is not necessarily projective, but its quotient module $M\otimes_R Q$ is projective as a right Q-module. Therefore, in the case where Q is non-semi-simple, it is interesting to obtain a condition under which finitely generated torsion free modules have projective quotient modules. The next proposition of this paper gives such a condition.

Proposition 1. *If a ring R has a two-sided perfect two-sided quotient ring Q, then the following conditions on a finitely generated right R-module M are equivalent*:

(1) *M is R-torsion free* (*in the sense of Levy* [5]) *and $M\otimes_R Q$ is Q-projective.*

(2) *M is isomorphic to a direct summand of a right R-module K such that $\sum_{i=1}^{n}\oplus R^{(i)}\supseteq K\supseteq \sum_{i=1}^{n}\oplus I_i$, where $R^{(i)}$ is a copy of R and I_i is a right ideal of $R^{(i)}$ containing a regular element.*

In this paper this condition (2), without assuming that M is finitely generated, will be called condition (A).

Then, we obtain the next main theorem which generalizes the above results of Small [7, Corollary 2] and Hart [3, Theorem 2].

Theorem 1. *If R is a ring with a right* (*resp. two-sided*) *Artinian right* (*resp. two-sided*) *quotient ring Q, then the endomorphism ring $\mathrm{End}_R(M)$ of a right R-module M satisfying condition* (A) *has also a right* (*resp. two-sided*) *Artinian right* (*resp. two-sided*) *quotient ring isomorphic to $\mathrm{End}_Q(M\otimes_R Q)=\mathrm{End}_R(M\otimes_R Q)$.*

As an application of Theorem 1, we shall prove finally

Theorem 2. *In Theorem 1, if Q is quasi-Frobenius and M is faithful, then $\mathrm{End}_R(M)$ has a quasi-Frobenius quotient ring which is isomorphic to the R-endomorphism ring of the injective hull of M.*

This result implies *the Morita-invariance* of "right (resp. two-sided) order in quasi-Frobenius ring".

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1. Preliminaries. Let R be a ring. An element $s \in R$ is said to be right (resp. left) regular if s has no non-zero right (resp. left) annihilator. A right and left regular element is called regular. A right R-module M is said to be torsion free, if no non-zero element of M is annihilated by a regular element of R. Levy [5, Proposition 1.5] proved that if M is torsion free right R-module and R has a right quotient ring Q, then the correspondence $\sigma: M \to M \otimes_R Q$ defined by $\sigma(m) = m \otimes 1$, where $m \in M$, is an R-monomorphism and every right Q-module M' such that it contains M as an R-submodule and can be represented by $M' = MQ$ is Q-isomorphic to $M \otimes_R Q$. In the following such a module MQ is called a quotient Q-module of M. Let M ond N be torsion free right R-modules. Then every $f \in \mathrm{Hom}_R(M, N)$ can be extended naturally to $f' \in \mathrm{Hom}_Q(MQ, NQ)$ by $f'(ms^{-1}) = f(m)s^{-1}$, where $m \in M$, $s \in R$ and s regular. Clearly this is the unique extension of f and it is easy to check $\mathrm{Hom}_Q(MQ, NQ) = \mathrm{Hom}_R(MQ, NQ)$. Then the endomorphism ring $\mathrm{End}_R(M)$ is regarded to be a subring of $\mathrm{End}_Q(MQ)$ $= \mathrm{End}_R(MQ)$.

Throughout this paper we assume that R is a ring (not necessarily with unity) which contains a regular element and that all homomorphisms between modules will be written on the left.

2. To begin with we shall prove Proposition 1.

Proof of Proposition 1. (1)$\Rightarrow$(2). There is a finitely generated right Q-module N' such that $MQ \oplus N' = \sum_{i=1}^{n} \oplus Q^{(i)}$, where MQ is the quotient Q-module of M and $Q^{(i)}$ is a copy of Q. Hence we can choose a finitely generated torsion free right R-module N such that $N' = NQ$. Now write $K = M \oplus N$, then $KQ = MQ \oplus NQ$, since M and N are R-torsion free. Let $e_1, \cdots, e_n$ be free Q-basis of KQ $\left(= \sum_{i=1}^{n} \oplus Q^{(i)}\right)$ and $k_1, \cdots, k_t$ be the R-generators of K. Write $k_i = \sum_{j=1}^{n} e_j p_{ij}$, $p_{ij} \in Q$, $(i = 1, \cdots, t)$. As Q is the left quotient ring of R, there exist a_{ij}, $d \in R$ and d regular such that $p_{ij} = d^{-1}a_{ij}$, $(i = 1, \cdots, t,\ j = 1, \cdots, n)$. Hence $K \subseteq \sum_{ij} e_j p_{ij} R + \sum_{ij} e_j P_{ij} Z \subseteq \sum_{j=1}^{n} \oplus e_j d^{-1} R$, where Z is the ring of rational integers. On the other hand, since Q is the right quotient ring of R and KQ is the quotient Q-module of K, there exist $x_i \in K$, $q_i \in Q$ such that $x_i q_i = e_i d^{-1}$, $(i = 1, \cdots, n)$. Put $x_i = \sum_{j=1}^{n} e_j d^{-1} r_{ij}$. Then we have $r_{ij} q_i = 0$ $(i \neq j)$, $r_{ii} q_i = 1$. It is to be noted that every right (resp. left) regular element in a left (resp. right) perfect ring with unit is invertible and then regular (cf. Gupta [2]). Hence r_{ii} and q_i are regular in Q and $r_{ij}=0$ $(i\neq j)$. Then $x_i=e_i d^{-1} r_{ii}$, where $r_{ii}\in R$. Let I_i be a right ideal of R generated by r_{ii} $(i=1,\cdots,n)$. Then we can conclude that $\sum_{i=1}^{n}\oplus e_i d^{-1}R\supseteq K\supseteq\sum_{i=1}^{n}\oplus e_i d^{-1}I_i$. This completes the proof of (1)$\Rightarrow$(2).

(2)$\Rightarrow$(1). Clearly M is R-torsion free. Write $M\oplus N=K$, where N is a submodule of K. Then $KQ=MQ\oplus NQ$. Since I_i contains a regular element, we have $KQ=\sum_{i=1}^{n}\oplus Q^{(i)}$.

Proposition 2. *If R has a right quotient ring Q and M is an unital right Q-module, then M_Q is Q-injective if and only if M_R is R-injective.*

Proof. If M_R is R-injective, then the proof of Gupta [1, Theorem 2.1] is valid even if R has not identity and hence M_Q is Q-injective. Conversely let M_Q be Q-injective. To prove that M_R is R-injective, it suffices to show that if B_R is a submodule of a right R-module A_R and $f: B_R\rightarrow M_R$ is an R-monomorphism, then f can be extended to $g: A_R\rightarrow M_R$, for we can always shift from any homomorphism $h: B_R\rightarrow M_R$ to the induced monomorphism $\bar{h}: [B/\ker(h)]_R\rightarrow M_R$. Since it is easy to see that M_R is R-torsion free, B_R is R-torsion free. Then f can be extended to $f': [BQ]_Q\rightarrow M_Q$, where $f'(bs^{-1})=f(b)s^{-1}$, $b\in B$, $s\in R$ and s regular. As was pointed out by Jans [4], $T(A)=\{a\in A\mid \text{there exists } s\in R \text{ regular such that } as=0\}$ is an R-submodule of A. Since B_R and $[A/T(A)]_R$ are R-torsion free, then $B\cap T(A)=0$, and B_R can be regarded to be an R-submodule of $[A/T(A)]_R$. As the quotient Q-module of B_R is determined uniquely up to isomorphism, $[BQ]_Q$ becomes a Q-submodule of $[\{A/T(A)\}Q]_Q$. Then f' can be extended to Q-homomorphism $g': [\{A/T(A)\}Q]_Q\rightarrow M_Q$, since M_Q is Q-injective. Let $\pi: A_R\rightarrow[A/T(A)]_R$ be the cannonical R-homomorphism, then $g=g'\pi: A_R\rightarrow M_R$ is the desired extension of f.

Let R be a ring with a right quotient ring Q. Let $Q^{(i)}$ and $R^{(i)}$ be copies of Q and R respectively and $R^{(i)}\subseteq Q^{(i)}$ $(i=1,\cdots,n)$. Then a right R-submodule M of $\sum_{i=1}^{n}\oplus R^{(i)}$ can be imbedded into $\sum_{i=1}^{n}\oplus Q^{(i)}$, and we can define a right Q-submodule MQ in a natural way. In connexion with Proposition 1 we have the next lemma which generalizes the result of Hart [3, Theorem 1], in his case M is assumed to be finitely generated and Q is a two-sided quotient ring.

Lemma 1. *Let R be a ring with a right (resp. two-sided) quotient ring Q and M a right R-submodule of $\sum_{i=1}^{n}\oplus R^{(i)}$ such that $MQ=\sum_{i=1}^{n}\oplus Q^{(i)}$. If every right regular (resp. regular) element of* $\mathrm{End}_Q(MQ)$ *is invert-*

ible, then $\mathrm{End}_R(M)$ *has a right (resp. two-sided) quotient ring isomorphic to* $\mathrm{End}_Q(MQ)$.

Proof. Let $\alpha \in \mathrm{End}_Q(MQ)$ be arbitrary. Choose the Q-free basis of MQ $e_1, \cdots, e_n$, where e_i has 1 in i-th component and 0 elsewhere $(i=1, \cdots, n)$. Put $\alpha(e_i)=\sum_{j=1}^{n} e_j q_{ij}\ (q_{ij} \in Q)$. First we shall consider the case where Q is the right quotient ring. Then there exists an element $s \in R$ regular such that $e_i s \in M$ and $e_j q_{ij} s \in M$ $(i, j=1, \cdots, n)$. Define $f' \in \mathrm{End}_Q(MQ)$ by $f'(z)=\sum_{i=1}^{n} e_i s p_i$, where $z=\sum_{i=1}^{n} e_i p_i \in MQ$ $(p_i \in Q)$. Then the restriction $f=f' \mid M$ is in $\mathrm{End}_R(M)$, since $z \in M$ implies $p_i \in R$. If $z \in M$, then $\alpha f'(z)=\sum_{i,j=1}^{n} e_j q_{ij} s p_i \in M$. Hence $g'=\alpha f'$ can be regarded to be an element of $\mathrm{End}_R(M)$. Since s is invertible in Q, f' is invertible $((f')^{-1}: z \to \sum_{i=1}^{n} e_i s^{-1} p_i)$. Therefore $\alpha=g'(f')^{-1}$. Next let $h \in \mathrm{End}_R(M)$ be regular and $h' \in \mathrm{End}_Q(MQ)$ the extension of h. Assume $h'\beta=0$ for any $\beta \in \mathrm{End}_Q(MQ)$. Then β can be written by $g_1'(f_1')^{-1}$, where g_1', $f_1' \in \mathrm{End}_Q(MQ)$ and $g_1=g_1' \mid M$, $f_1=f_1' \mid M$ are elements of $\mathrm{End}_R(M)$. Therefore $h'\beta=h'g_1'(f_1')^{-1}=0$, $h'g_1'=0$ and $hg_1=0$. Since h is regular in $\mathrm{End}_R(M)$, it follows that $g_1=0$ and $g_1'=0$. This implies $\beta=0$ and h' is right regular in $\mathrm{End}_Q(MQ)$. Hence $\mathrm{End}_Q(MQ)$ is the right quotient ring of $\mathrm{End}_R(M)$.

Next we shall consider the case where Q is the two-sided quotient ring. We can choose regular elements t, $u \in R$ such that $e_i t \in M$, $u q_{ij} \in R$ $(i, j=1, \cdots, n)$. Define $f'' \in \mathrm{End}_Q(MQ)$ by putting $f''(z) = \sum_{i=1}^{n} e_i t u p_i$, where $z=\sum_{i=1}^{n} e_i p_i \in MQ$, $(p_i \in Q)$. Then the restriction $f'' \mid M$ is in $\mathrm{End}_R(M)$. If $z \in M$, then $f''\alpha(z)=\sum_{i,j=1}^{n} e_j t u q_{ij} p_i \in M$. Then $g''=f''\alpha$ can be regarded to be an element of $\mathrm{End}_R(M)$. Hence $\alpha=(f'')^{-1}g''$. As similarly as in the first case every regular element of $\mathrm{End}_R(M)$ is regular in $\mathrm{End}_Q(MQ)$.

Now we are able to prove Theorem 1.

Proof of Theorem 1. If $K=M \oplus N$ where N is a submodule of K, then $KQ=MQ \oplus NQ=\sum_{i=1}^{n} \oplus Q^{(i)}$. Since $\mathrm{End}_Q(KQ) \cong Q_n$, the complete ring of $n \times n$ matrices over Q, it is a right (resp. two-sided) Artinian ring. Hence by Lemma 1, $\mathrm{End}_Q(KQ)$ is a right (resp. two-sided) quotient ring of $\mathrm{End}_R(K)$. Let $e \in \mathrm{End}_R(K)$ be the projection $M \oplus N \to M$. Since e is an idempotent element, by Small [7, Theorem 3] and its right left symmetry we can deduce that $e'(\mathrm{End}_Q(KQ))e'$ is a right (resp. two-sided) Artinian right (resp. two-sided) quotient ring of $e(\mathrm{End}_R(K))e$, where $e' \in \mathrm{End}_Q(KQ)$ is the extension of e. Since e' is the projection $MQ \oplus NQ \to MQ$, we have $e(\mathrm{End}_R(K))e \cong \mathrm{End}_R(M)$ and $e'(\mathrm{End}_Q(KQ))e' \cong \mathrm{End}_Q(MQ)$. This completes the proof.

Remark. Let R be a semi-prime right Goldie ring. Then every non-zero right ideal satisfies condition (A), for it is a direct summand of an essential right ideal containing a regular element. Therefore, several results by Hart [3], e.g. Theorems 4-6 are obtained immediately from our Theorem 1.

Proof of Theorem 2. From the assumption, we can easily check that $[MQ]_Q$ is a faithful, finitely generated and projective Q-module. By the result of Morita [6, Theorem 16.6], $\mathrm{End}_Q(MQ)$ is quasi-Frobenius. Since Q is quasi-Frobenius, $[MQ]_Q$ is an injective Q-module and $[MQ]_R$ is an injective R-module by Proposition 2. On the other hand $[MQ]_R$ is an essential extension of M_R. Hence it is the injective hull of M_R.

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