64. *On Semi-inner Product Algebras**)

By T. HUSAIN and B. D. MALVIYA
Department of Mathematics, McMaster University, Hamilton, Canada

(Comm. by Kinjirô KUNUGI, M. J. A., March 12, 1970)

1. Introduction. R. Keown [5] introduced some new classes of commutative Hilbert algebras which is some sense are generalizations of the algebras studied by W. Ambrose [1]. The essential difference between the works of Keown and Ambrose is that the latter doe not obtain the decomposition of the algebra into orthogonal subspaces each of which is a minimal left ideal. The present authors [4] generalized the work of Ambrose by replacing the underlying Hilbert space structure by a more general space called the semi-inner product space, a concept introduced by G. Lumer [6]. The purpose of this note is to extend some of Keown's results to semi-inner product spaces (henceforth abbreviated to s.i.p. spaces). For example, we show that for any generalized s.i.p. algebra A and for an idempotent e, eAe is a division algebra. For definitions we follow Keown [5] and Husain [2].

2. We recall some of the definitions from [4] and [6].

A complex (real) vector space X is called a *complex* (real) s.i.p. space if corresponding to any pair of elements $x, y \in X$, there is defined a complex (real) number $[x, y]$ which satisfies the following properties:

(i) $[x+y, z]=[x, z]+[y, z]$,
$[\lambda x, y]=\lambda[x, y]$ for $x, y, z \in X$, λ is complex or real,

(ii) $[x, x]>0$ for $x\neq 0$,

(iii) $|[x, y]|^2\leq [x, x][y, y]$.

We put $\|x\|=[x, x]^{1/2}$ and thus X is a normed space. However an s.i.p. space need *not* satisfy the following properties:

(iv) $[x, \lambda y]=\bar{\lambda}[x, y]$,

(iv)′ $[x, y]=[y, x]$

(v) $[x, y+z]=[x, y]+[x, z]$.

A s.i.p. X space is said to be *continuous* if

$$\mathrm{Re}\,\{[y, x+\lambda y]\}\rightarrow \mathrm{Re}\,\{[y, x]\} \qquad \text{for all real } \lambda\rightarrow 0,$$

and any $x, y \in X$. In a s.i.p. space X, an element $x \in X$ is said to be *orthogonal* to $y \in X$ if $[y, x]=0$. A s.i.p. space is said to be *strictly convex* if $\|x+y\|=\|x\|+\|y\|$ implies $y=\lambda x$, $\lambda>0$. An s.i.p. space which is also a Banach algebra is said to be a *generalized s.i.p. algebra.* A generalized s.i.p. algebra A is said to be *regular* if corresponding

*) The work was supported by the National Research Council of Canada.

to every maximal modular ideal R of A, there is an ideal I such that $A=R+I$. A generalized s.i.p. algebra is said to be *proper* if it contains non-zero annihilators. An idempotent e is said to be *primitive*, if e can not be expressed as $e=e_1+e_2$, where $0\neq e_1, e_2$ are idempotents. A generalized s.i.p. algebra A is said to be *adjoint* if there exist binary operations $f:(x,y)\to xy$ and $f^*:(x,y)\to x\cdot y$ such that $(xy,z)=(y,x^*\cdot z)$, for any $x,y,z\in A$.

The following lemmas from [3] are needed in the sequel. We quote them here without proofs.

Lemma 1. *Let X be a complete and continuous s.i.p. space which satisfies the inequality $\|u+v\|^2+\mu^2\|u-v\|^2\leq 2\|u\|^2+2\|v\|^2$, $(0<\mu<1)$, for all $u,v\in X$, then there is a non-zero vector orthogonal to a closed proper subspace Y of X and any $x\in X$ can be expressed in the form: $x=y+z$, where y belongs to Y and Z is orthogonal to Y.*

Lemma 2. *In a continuous s.i.p. space X which is complete with respect to its norm and in addition the norm satisfies the inequality: $\|u+v\|^2+\mu^2\|u-v\|^2\leq 2\|u\|^2+2\|v\|^2$, $(0<\mu<1)$, every continuous linear functional f defined on X can be represented by $f(x)=[x,y]$, where y is unique.*

Lemma 3. *In the case of weak convergence with respect to the second argument of the semi-inner product the weak limit is unique if the s.i.p. space is strictly convex.*

A less general form of result contained in Husain and Malviya [4] is also needed in the sequel. In Lemmas 4 and 5 below, the generalized proper s.i.p. algebra A has the involution * defined by $[xy,z]=[y,x^*\cdot z]$, where $x,y,z\in A$. In addition the involution is taken to satisfy the condition: $(\alpha x+\beta y)=\bar{\alpha}x^*+\bar{\beta}y^*$, where α,β are scalars. We also assume that the norm in the s.i.p. space satisfies the inequality:

$$\|u+v\|^2+\mu^2\|u-v\|^2\leq 2\|u\|^2+2\|v\|^2,\ 0<\mu<1.$$

Lemma 4. *A generalized s.i.p. algebra A with involution satisfying the conditions above, contains primitive idempotents.*

Lemma 5. *In a generalized s.i.p. algebra A with involution defined as above and satisfying the condition $[x,y]=[y^*,x^*]$ for any $x,y\in A$; the right ideal $R=eA$ is minimal if and only if e is a primitive idempotent. (The same is true for left ideal also.)*

3. As in Keown [5] we take all the algebras to be commutative and semi-simple in the sequel. Furthermore we assume that the norm in the s.i.p. space (complete and continuous) satisfies the inequality

$$\|u+v\|^2+\mu^2\|u-v\|^2\leq 2\|u\|^2+2\|v\|^2, \qquad (0<\mu<1).$$

Lemma 1 can be used to adopt Keown's proof of (cf: [5], Lemma 2.1) of the following result for the Hilbert regular algebras to generalized s.i.p. regular algebras with appropriate changes.

Lemma 6. *Every maximal modular ideal R of a generalized s.i.p. regular algebra A has associated with it a unique minimal idempotent e and a unique multiplicative element g. An element $x \in A$ is in R iff either $ex=0$ or $[x, g]=0$. For any $x, y \in A$, $[xy, g]=[x, g][y, g]$.*

First we show that under the topology induced by the multiplicative linear functionals on a generalized s.i.p. regular algebra, the algebra is a topological algebra. More precisely, we prove the following:

Proposition 1. *With respect to the topology induced by the multiplicative linear functionals, the generalized s.i.p. regular algebra A is a topological algebra.*

Proof. The topology is generated by taking the collection of all subsets of A of the form $\{x : |f_i(x-x_0)|<\varepsilon, x_0 \in A, i=1, 2, \cdots, n\}$ as an open sub-basis for the topology on A where f_i's are the multiplicative linear functionals (as obtained in Lemma 6 by putting $f(x)=[x, g]$) defined over A. It is easy to check that $(x, y)\to x+y$ and $(\lambda, x)\to \lambda x$ are continuous with respect to this topology. To show that $(x, y)\to xy$ is continuous under this topology, let $x_0, y_0 \in A, f_i(xy-x_0y_0)=f_i\{x(y-y_0)+(x-x_0)y_0\}=f_i(x)f_i(y-y_0)+f_i(x-x_0)f_i(y_0)$. Consider a sub-basic neighbourhood of x_0y_0 defined by $\{z : |f_i(z-x_0y_0)|<\varepsilon, i=1, 2, \cdots, n\}$. For $\delta>0$, let $x \in \{p : |f_i(p-x_0)|<\delta\}$ and $y \in \{q : |f_i(q-y_0)|<\delta\}$. We have $|f_i(x)|\le|f_i(x-x_0)|+|f_i(x_0)|<\delta+|f_i(x_0)|$. So $|f_i(xy-x_0y_0)|\le\{\delta+|f_i(x_0)|\}\delta+\delta|f_i(y_0)|\le\varepsilon$, provided we choose δ small enough. Thus $xy \in \{z : |f_i(z-x_0y_0)|<\varepsilon\}$.

In view of Lemma 3.1 ([5]), the following is clear:

Lemma 7. *The orthogonal complement of a standard (adjoint) ideal I of the adjoint algebra A is an adjoint (standard) ideal J of A provided that the s.i.p. space has the property* (v) *in* § 2.

Lemma 8. *A generalized s.i.p. adjoint algebra A is regular under the standard and adjoint products separately provided the s.i.p. space satisfies the property* (v) *in* § 2.

Proof. Let R be a standard maximal modular ideal of A. Then by Lemma 1, $A=R+J$ (orthogonal sum). Also by Lemma 7, J is an adjoint ideal. Now the proof is completed by following Keown ([5], Lemma 3.2).

Lemma 9. *In a finite dimensional generalized s.i.p. adjoint algebra the standard (adjoint) socle is dense in A.*

The proof is the same as in Keown ([5] Lemma 3.3). The finite dimensionality is needed to ensure the denumerability of the minimal idempotents in the generalized s.i.p. adjoint algebra.

Proposition 2. *Let A be a finite dimensional generalized s.i.p. adjoint algebra and * an involution, then $[x, y]=[y^*, x^*]$, assuming* *that the s.i.p. space satisfies the property* (iv) *in* § 2 *and the strong convergence of a sequence from the socle implies weak convergence with respect to the second argument of the semi-inner product.*

Proof. We have for $x \in A$ and any minimal idempotent e_i, $[e_i, x] = [e_i^2, x] = [e_i, e_i^* \cdot x] = [e_i x^*, e_i^*] = [x^*, e_i^* \cdot e_i^*] = [x^*, e_i^*]$, since $(xy)^* = y^* \cdot x^*$. Now consider the element z_n of the socle, then $z_n = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n$; and let y be any other element of A. Then by (iv) in § 2 and the above relation we have $[z_n, y] = [y^*, z_n^*]$. Taking $y = z_n$, we have $\|z_n\| = \|z_n^*\|$. This implies the continuity of involution. The continuity of the involution in essence means that $z_n \to z \Rightarrow z_n^* \to z^*$. Now we have $\lim [z_n, y] = \lim [y^*, z_n^*]$ or $[z, y] = \lim [y^*, z_n^*]$. But since the s.i.p. space satisfies the inequality $\|u+v\|^2 + \mu^2 \|u-v\|^2 \le 2\|u\|^2 + 2\|v\|^2$, this implies that the space is uniformly convex and hence also strictly convex. Hence by Lemmas 3 and 9; and using the fact that socle is dense, the result follows.

From the definition it is clear that $(xy)^* = y^* \cdot x^* = x^* \cdot y^*$. In addition we assume that involution in A satisfies $(\alpha x + \beta y)^* = \bar{\alpha} x^* + \bar{\beta} y^*$ (α, β are scalars), since this does not follow from the definition of * in the s.i.p. space because of the lack of linearly in the second argument. We now prove the following proposition:

Proposition 3. *In a generalized proper s.i.p. adjoint algebra A, eAe is a division algebra where e is an idempotent, assuming that A satisfies the conditions of Proposition* 2.

Proof. From Lemma 4, it is seen that e is primitive. By Lemma 5, Ae is a minimal ideal. Let $0 \neq x \in A$, then $exe \in eAe$. Now $Aexe \subset Ae$. Since $Aexe \neq 0$, hence $Aexe = Ae$. Again $e \in Ae \Rightarrow e \in Aexe$. Hence for some $y \in A$, $yexe = e$. Then $(eye)(exe) = e$. Let $z = exe\ eye$, then $z^2 = z$. Also $z \neq 0$, for if $(exe)(eye) = 0$ then $0 = (exe)(eye)(exe) = (exe)e = exe \neq 0$. This shows that z is an idempotent in eAe. Since e is primitive, it is the only idempotent in eAe, therefore $z = e$. Thus every non-zero element of eAe has an inverse and eAe is a division algebra.

References

[1] W. Ambrose: Structure theorems for a special class of Banach algebras. Trans. Amer. Math. Soc., **57**, 366–386 (1945).

[2] T. Husain: Introduction to Topological Groups. W. B. Saunder Company (1966).

[3] T. Husain and B. D. Malviya: On semi-inner product spaces. I (to appear).

[4] ——: On semi-inner product spaces. II (to appear).

[5] R. Keown: Some new Hilbert algebras. Trans. Amer. Math. Soc., **128**, 71–87 (1967).

[6] G. Lumer: Semi-inner product spaces. Trans. Amer. Math. Soc., **100**, 29–43 (1961).