62. *Asymptotic Property of Solutions of Some Higher Order Hyperbolic Equations.* I

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Introduction. Let X be a complex Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let L be a selfadjoint (in general unbounded) operator on X satisfying

(1) $$(Lf, f)\geq 0 \qquad \text{for all } f\in \mathcal{D}(L),$$

where $\mathcal{D}(L)$ denotes the domain of L. We shall consider abstract "hyperbolic" equations of the form

(2) $$\prod_{j=1}^{m}[\partial_t^2+\alpha_j L]u(t)=0 \ (t\in \boldsymbol{R}^1)$$

$(\partial_t = d/dt)$ with initial data

(3) $$\partial_t^{j-1}u|_{t=0}=\varphi_j\in\mathcal{D}(L^{(2m-j+1)/2}),\ j=1,2,\cdots,2m,$$

where m is a positive integer and α_j are positive constants such that

(4) $$0<\alpha_1<\alpha_2<\cdots<\alpha_m.$$

In Mizohata [2], we know that there exists a unique solution of (2), (3) in the class $\bigcap_{0\leq j\leq 2m}\mathcal{E}_t^j(\mathcal{D}(L^{(2m-j)/2}))$[1] ([2]; Theorem 5.1). In this note, we shall obtain an asymptotic property as $t\to\infty$ of the solution under the assumption that the spectrum of L is strongly absolutely continuous with respect to the Lebesgue measure. As will be seen, we shall generalize recent results of Shinbrot [4] and Goldstein [1], in which are treated the case of abstract wave equations (i.e., when $m=1$ in (2)).

First we consider the case when the origin 0 is in the resolvent set of L. In this case, applying the method developed by Mizohata [2], we can construct the explicit formula of the strongly continuous group $\{T_t;\, t\in \boldsymbol{R}^1\}$ of unitary operators in the space $\prod_{j=1}^{2m}\mathcal{D}(L^{(2m-j)/2})$ which assign to given initial data $(\varphi_1,\varphi_2,\cdots,\varphi_{2m})$ the data of corresponding solution of (2) at time t. For the general case, let $L_n = L+2n^{-1}L^{1/2}+n^{-2}I$. Then, by the limit procedure developed by Goldstein [1], we can deduce the general case from the special case that L is invertible.

1. Assume first that there exists a positive constant c such that

(5) $$(Lf, f)\geq c\|f\|^2 \qquad \text{for all } f\in\mathcal{D}(L).$$

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1) $u(t)\in\mathcal{E}_t^j(X)$ means that $u(t)$ is j times continuously differentiable in t with values in X.

We put $H=L^{1/2}$. Then for each $j\geq 0$ integer, $\mathcal{D}(H^j)$ is a linear subspace of X, and we have

(6) $$\|Hf\|\geq\sqrt{c}\,\|f\| \qquad \text{for all } f\in\mathcal{D}(H).$$

Equation (2) can be written in the form

(7) $$\partial_t^{2m}u+\beta_1 L\partial_t^{2m-2}u+\cdots+\beta_m L^m u=0.$$

We put

(8) $$u_1=u,\ u_2=\partial_t u,\ \cdots,\ u_{2m}=\partial_t^{2m-1}u.$$

Then it follows from (7) that

$$\partial_t\begin{bmatrix}u_1\\ u_2\\ \vdots\\ u_{2m}\end{bmatrix}=\begin{bmatrix}0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & \ddots & \ddots & & & \\ & & & & & 0 & 1\\ -\beta_m L^m & 0 & -\beta_{m-1}L^{m-1} & 0 & \cdots & -\beta_1 L & 0\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ \vdots\\ u_{2m}\end{bmatrix}.$$

We write this simply as

(9) $$\partial_t U(t)=AU(t), \qquad U={}^t(u_1,u_2,\cdots,u_{2m}).^{2)}$$

This equation will be considered as a differential equation in the space $\mathcal{D}(H^{2m-1})\times\mathcal{D}(H^{2m-2})\times\cdots\times\mathcal{D}(H^0)=\prod_{j=1}^{2m}\mathcal{D}(H^{2m-j})$, where the domain of A is given as $\mathcal{D}(A)=\prod_{j=1}^{2m}\mathcal{D}(H^{2m-j+1})$.

We put $X_j=\mathcal{D}(H^j)$ $(X_0=X)$. Then each X_j forms a Hilbert space with norm

$$\|f\|_j=\|H^j f\|, \qquad f\in X_j.$$

Thus, in $\prod_{j=1}^{2m}X_{2m-j}$ is defined the naturally induced norm

$$\|F\|_{\tilde{\mathcal{H}}}=\Big[\sum_{j=1}^{2m}\|f_j\|_{2m-j}^2\Big]^{1/2}, \quad F={}^t(f_1,f_2,\cdots,f_{2m}).$$

However, we define another norm (energy norm) in this space (cf., Mizohata [2]).

We introduce the matrix

(10) $$E(H)=\begin{bmatrix}H^{2m-1} & & & \\ & H^{2m-2} & & \\ & & \ddots & \\ & & & H^0\end{bmatrix}.$$

$E(H)$ maps $\prod_{j=1}^{2m}X_{2m-j}$ one-to-one onto X^{2m}, and it follows that

(11) $$E(H)A=PHE(H),$$

where

$$P=\begin{bmatrix}0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & \ddots & \ddots & & & \\ & & & & & 0 & 1\\ -\beta_m & 0 & -\beta_{m-1} & 0 & \cdots & -\beta_1 & 0\end{bmatrix}.$$

Since the equation $\det[\gamma I-P]=0$ has the distinct roots $\gamma=\pm\sqrt{\alpha_j}$

2) If M is matrix, ${}^tM$ denotes the transpose of M.

$(j=1,2,\cdots,m)$ by (4), there exists a non-singular matrix $N=(n_{jk})$ such that

(12) $$NP=iDN \quad (i=\sqrt{-1}),$$

where

$$D=\begin{bmatrix} +\sqrt{\alpha_1} & & & & \\ & -\sqrt{\alpha_1} & & & \\ & & \ddots & & \\ & & & +\sqrt{\alpha_m} & \\ & & & & -\sqrt{\alpha_m} \end{bmatrix}.$$

We introduce the following notation:

(13) $$\gamma_{2j-1}=+i\sqrt{\alpha_j} \text{ and } \gamma_{2j}=-i\sqrt{\alpha_j} \quad (j=1,2,\cdots,m).$$

Then N^{-1} is given as follows:

$$N^{-1}=\begin{bmatrix} 1 & 1 & \cdots\cdots & 1 \\ \gamma_1 & \gamma_2 & \cdots\cdots & \gamma_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \gamma_1^{2m-1} & \gamma_2^{2m-1} & \cdots & \gamma_{2m}^{2m-1} \end{bmatrix}.$$

Now we define in the space $\prod_{j=1}^{2m} X_{2m-j}$ the following new inner product

$$\begin{aligned}(F,G)_{\mathcal{H}}&=(NE(H)F,NE(H)G)_{X^{2m}}\\ &=\sum_{j=1}^{2m}\left(\sum_{k=1}^{2m} n_{jk}H^{2m-k}f_k,\ \sum_{k=1}^{2m} n_{jk}H^{2m-k}g_k\right).\end{aligned}$$

Then $\|F\|_{\mathcal{H}}=(F,F)_{\mathcal{H}}^{1/2}$ is equivalent to the $\tilde{\mathcal{H}}$-norm. We denote by $\mathcal{H}$ the Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{H}}$ and norm $\|\cdot\|_{\mathcal{H}}$.

Theorem 1. *The operator A, with domain $\mathcal{D}(A)=\prod_{j=1}^{2m}\mathcal{D}(H^{2m-j+1})$, is skew selfadjoint in $\mathcal{H}$.*

Proof. From (11) and (12), it follows that

(14) $$(AF,G)_{\mathcal{H}}=(iDHNE(H)F,NE(H)G)_{X^{2m}}$$

for any $F\in\mathcal{D}(A)$ and $G\in\mathcal{H}$. Note that $E(H)\mathcal{H}=X^{2m}$ and $E(H)\mathcal{D}(A)=(\mathcal{D}(H))^{2m}$. Then since DH is selfadjoint in X^{2m} with domain $(\mathcal{D}(H))^{2m}$, we see from (14) that $A^*=-A$. q.e.d.

It now follows that A generates a strongly continuous group $\{T_t=e^{At};t\in\boldsymbol{R}^1\}$ of unitary operator in $\mathcal{H}$ with the following properties:

(a) T_tF is strongly differentiable in t if and only if F belongs to $\mathcal{D}(A)$, in which case

(15) $$\partial_t T_t F=AT_tF,$$

(b) T_t maps $\mathcal{D}(A)$ onto $\mathcal{D}(A)$ and commutes with A.

Suppose that $F\in\mathcal{D}(A)$, and denote the first component of T_tF by $u(t)$. Then $u(t)\in\mathcal{D}(H^{2m})=\mathcal{D}(L^m)$ and the last component of relation (15) gives

(16) $$\partial_t^{2m}u=-\beta_mL^mu-\beta_{m-1}L^{m-1}\partial_t^2u-\cdots-\beta_1L\partial_t^{2(m-1)}u;$$

that is, $u(t)$ satisfies equation (2).

From (14), it is not difficult to verify the following two lemmas.

Lemma 1. *The j-th component of $T_tF(F \in \mathcal{H})$ is expressed as*

$$(17) \qquad [T_tF]_j = \sum_{k=1}^{2m} (\gamma_k)^{j-1} e^{\gamma_k Ht} \sum_{l=1}^{2m} n_{kl} H^{j-l} f_l.$$

Lemma 2. *Let p be any integer such that $p \leq 2m$, and let*

$$(18) \qquad \Gamma^F_{p,j} = \left\| \sum_{k=1}^{2m} n_{jk} H^{p-k} f_k \right\|^2.$$

Then

$$(19) \qquad \left\| \sum_{k=1}^{2m} n_{jk} H^{p-k} [T_tF]_k \right\|^2 = \Gamma^F_{p,j} \quad \text{for all } t \in \boldsymbol{R}^1.$$

We can now prove the following theorem.

Theorem 2. *Let L be a selfadjoint operator in X satisfying* (5). *Suppose that the spectrum of L is strongly absolutely continuous with respect to the Lebesgue measure. Then for any $\Phi = {}^t(\varphi_1, \varphi_2, \cdots, \varphi_{2m}) \in \mathcal{D}(L^m) \times \mathcal{D}(L^{(2m-1)/2}) \times \cdots \times \mathcal{D}(L^{1/2})$, the solution $u(t) = [T_t\Phi]_1$ of* (2), (3) *has the following asymptotic properties:*

$$(20) \qquad \lim_{t\to\infty} \|H^{p-j}\partial_t^{j-1}u(t)\|^2 = \sum_{k=1}^{2m} |\gamma_k|^{2(j-1)} \Gamma^{\Phi}_{p,k} \ (j=1, 2, \cdots, 2m),$$

where $H = L^{1/2}$ and p is any integer such that $p \leq 2m$.

Proof. Let $\{E^L_\sigma ; \sigma \in \boldsymbol{R}^1\}$ and $\{E^H_\sigma ; \sigma \in \boldsymbol{R}^1\}$ be the resolutions of the identity for L and H, respectively. Then since $E^H_\sigma = E^L_{\sigma^2}$ for all $\sigma \in \boldsymbol{R}^1_+ = (0, \infty)$, $\sigma \to E^H_\sigma f$ $(f \in X)$ is strongly absolutely continuous.

Put $\tilde{\varphi}_{j,p} = \sum_{k=1}^{2m} n_{jk} H^{p-k} \varphi_k$. Then noting (13), we have from (17)

$$H^{p-j}\partial_t^{j-1}u(t) = \sum_{k=1}^{m} (i\sqrt{\alpha_k})^{j-1} \{e^{i\sqrt{\alpha_k}Ht}\tilde{\varphi}_{2k-1,p} + (-1)^{j-1} e^{-i\sqrt{\alpha_k}Ht}\tilde{\varphi}_{2k,p}\}.$$

Thus

$$\|H^{p-j}\partial_t^{j-1}u(t)\|^2 = \sum_{k=1}^{m} \alpha_k^{j-1} \{\|\tilde{\varphi}_{2k-1,p}\|^2 + \|\tilde{\varphi}_{2k,p}\|^2\} + J(t),$$

where

$$\begin{aligned} J(t) = {} & 2\mathrm{Re} \sum_{k=1}^{m} (-1)^{j-1} \alpha_k^{j-1} (e^{i2\sqrt{\alpha_k}Ht}\tilde{\varphi}_{2k-1,p}, \tilde{\varphi}_{2k,p}) \\ & + 2\mathrm{Re} \sum_{l=1}^{m} \sum_{k<l} (\sqrt{\alpha_k \alpha_l})^{j-1} \{(e^{i(\sqrt{\alpha_k} - \sqrt{\alpha_l})Ht}\tilde{\varphi}_{2k-1,p}, \tilde{\varphi}_{2l-1,p}) \\ & + (\tilde{\varphi}_{2k,p}, e^{i(\sqrt{\alpha_k} - \sqrt{\alpha_l})Ht}\tilde{\varphi}_{2l,p}) + (-1)^{j-1}(e^{i(\sqrt{\alpha_k} + \sqrt{\alpha_l})Ht}\tilde{\varphi}_{2k-1,p}, \tilde{\varphi}_{2l,p}) \\ & + (-1)^{j-1}(\tilde{\varphi}_{2k,p}, e^{i(\sqrt{\alpha_k} + \sqrt{\alpha_l})Ht}\tilde{\varphi}_{2l-1,p})\}. \end{aligned}$$

For any $\gamma \neq 0$ real, $e^{i\gamma Ht}$ is represented as

$$(21) \qquad (e^{i\gamma Ht}f, g) = \int e^{i\gamma\sigma t} d(E^H_\sigma f, g) \quad \text{for } f, g \in X.$$

Since the scalar measure $\mathrm{dm}(\sigma) = d(E^H_\sigma f, g)$ is absolutely continuous, it follows from the Riemann-Lebesgue theorem that (21), which is the Fourier transform of $\mathrm{dm}(\sigma)$, tends as $t \to \infty$ to zero. Thus noting (4), we deduce that $\lim_{t\to\infty} J(t) = 0$. q.e.d.

Corollary 1. *In* (20), *if we put $p = j = 1$, then it follows that*

(20)′ $$\lim_{t\to\infty}\|u(t)\|^2=\sum_{k=1}^{2m}\Gamma_{1,k}^{\Phi}=\|H^{-2m+1}\Phi\|_{\mathcal{H}}^2.$$

2. Next, for the general case, we can prove the following theorem by the limit procedure (see Goldstein [1]).

Theorem 3. *Let L be a selfadjoint operator in X satisfying* (1). *Then for any $\Phi={}^t(\varphi_1,\varphi_2,\cdots,\varphi_{2m})\in\mathcal{D}(L^m)\times\mathcal{D}(L^{(2m-1)/2})\times\cdots\times\mathcal{D}(L^{1/2})$, the initial value problem* (2), (3) *has a unique solution in the class $\bigcap_{0\le j\le 2m}\mathcal{E}_t^j(\mathcal{D}(L^{(2m-j)/2}))$. Let $\Gamma_{2m,j}^{\Phi}=\left\|\sum_{k=1}^{2m}n_{jk}H^{2m-k}\varphi_k\right\|^2$. Then*

(22) $$\left\|\sum_{k=1}^{2m}n_{jk}H^{2m-k}\partial_t^{k-1}u(t)\right\|^2=\Gamma_{2m,j}^{\Phi}.$$

Moreover, if the spectrum of L is strongly absolutely continuous with respect to the Lebesgue measure, then

(23) $$\lim_{t\to\infty}\|H^{2m-j}\partial_t^{j-1}u(t)\|^2=\sum_{k=1}^{2m}|\gamma_k|^{2(j-1)}\Gamma_{2m,k}^{\Phi}\quad(j=1,2,\cdots,2m).$$

Proof. Let $L_n=L+2n^{-1}L^{1/2}+n^{-2}I$, so that $L_n^{1/2}\equiv H_n=H+n^{-1}I$ ($n>0$ integer). Let $u^{(n)}(t)$ be the unique solution of (2) with L replaced by L_n with initial data (3). Then as was shown previously

(23) $$\partial_t^{2m-1}u^{(n)}(t)=\sum_{k=1}^{2m}(\gamma_k)^{2m-1}e^{\gamma_kH_nt}\sum_{l=1}^{2m}n_{kl}H_n^{2m-l}\varphi_l.$$

Since

$$e^{\gamma_kH_nt}=e^{\gamma_kt/n}e^{\gamma_kHt},$$

as $n\to\infty$, $\partial_t^{2m-1}u^{(n)}(t)$ converges in X uniformly on compact intervals to a necessarily strongly continuous function $u_{2m}(t)\in\mathcal{D}(H)$ given by

(24) $$u_{2m}(t)=\sum_{k=1}^{2m}(\gamma_k)^{2m-1}e^{\gamma_kHt}\sum_{l=1}^{2m}n_{kl}H^{2m-l}\varphi_l.$$

Let us define the functions $u_j(t)$ ($j=1,2,\cdots,2m-1$) inductively as

$$u_j(t)=\int_0^t u_{j+1}(s)ds+\varphi_j.$$

Then $u_j(t)\in\mathcal{D}(H^{2m-j+1})$ and as $n\to\infty$

$$\partial_t^{j-1}u^{(n)}(t)=\int_0^t\partial_s^ju^{(n)}(s)ds+\varphi_j\to u_j(t)$$

uniformly on compact intervals. By definition $u_j(t)=\partial_t^{j-1}u_1(t)$ ($j=1,2,\cdots,2m$). Further since $\varphi_j\in\mathcal{D}(H^{2m-j+1})$, it follows from (24) that $u_{2m}(t)$ is strongly continuously differentiable and

(25) $$\partial_tu_{2m}(t)\equiv\partial_t^{2m}u_1(t)=\sum_{k=1}^{2m}(\gamma_k)^{2m}e^{\gamma_kHt}\sum_{l=1}^{2m}n_{kl}H^{2m-l+1}\varphi_l.$$

Since $\int_0^te^{\gamma_kHs}fds\in\mathcal{D}(H)$ for all $f\in X$ and $H\int_0^te^{\gamma_kHs}fds=\gamma_k^{-1}\{e^{\gamma_kHt}f-f\}$, it is not difficult to see, by induction, that

(26) $$H^{2m-j}\partial_t^{j-1}u_1(t)=\sum_{k=1}^{2m}(\gamma_k)^{j-1}e^{\gamma_kHt}\sum_{l=1}^{2m}n_{kl}H^{2m-l}\varphi_l$$

and

(27) $$H^{2m-j+1}\partial_t^{j-1}u_1(t)=\sum_{k=1}^{2m}(\gamma_k)^{j-1}e^{\gamma_kHt}\sum_{l=1}^{2m}n_{kl}H^{2m-l+1}\varphi_l.$$

Now it follows from (25) and (27) that

$$\partial_t^{2m}u_1(t)+\sum_{j=1}^{m}\beta_j L^j\partial_t^{2(m-j)}u_1(t)$$
$$=\sum_{k=1}^{2m}\{(\gamma_k)^{2m}+\sum_{j=1}^{m}\beta_j(\gamma_k)^{2(m-j)}\}e^{\gamma_k Ht}\sum_{l=1}^{2m}n_{kl}H^{2m-l+1}\varphi_l.$$

The right member is zero by (12). Hence $u_1(t)$ defined above satisfies (2) and (3). (22) follows immediately from (26). The uniqueness of solutions is a consequence of (22) and linearity. (23) also follows from (26) by the same argument as in the proof of Theorem 2. q.e.d.

Corollary 2. *In* (23), *if we put* $j=1$, *then it follows that*

$$\lim_{t\to\infty}\|H^{2m-1}u(t)\|^2=\sum_{k=1}^{2m}\Gamma_{2m,k}^{\Phi}=\|\Phi\|_{\mathcal{H}}^2. \tag{23$'$}$$

(References are listed at the end of the next article, pp. 271–272.)