52. *An Estimate from above for the Entropy and the Topological Entropy of a C^1-diffeomorphism*

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Let φ be a C^1-diffeomorphism from an n-dimensional Riemannian manifold on itself, $h(\varphi)$ the topological entropy [1] of φ and let λ be a contractive constant of φ. In this paper, we will give an estimate from above for the topological entropy:

$$h(\varphi) \leqslant n \log 1/\lambda$$

Using a result of L. Goodwyn [3], one can derive also an estimate from above for the measure theoretic entropy [7]:

$$h_\mu(\varphi) \leqslant n \log 1/\lambda$$

and this estimate is sharper than Kuchinirenko's [6] and A. Avez's [2].

§1. Definitions and a property.

Let φ be a homeomorphism from a compact metric space X onto itself. If α is any open cover of X, we let $N(\alpha)$ be the number of members in a subcover of α of minimal cardinality. As in [1], the limit exists in the following definition:

$$h(\alpha, \varphi) = \lim_{m\to\infty} \frac{1}{m} \log N(V_{i=0}^{m-1} \varphi^i \alpha)^{*)}$$

Let α_t be the collection of all open spheres of radius $t>0$. In metric spaces, the topological entropy $h(\varphi)$ of φ can be defined as $h(\varphi) = \lim_{t\to 0} h(\alpha_t, \varphi)$. (This is equivalent to the usual definition.)

For any $t>0$, let β_t be any cover of subset A of X by arbitrary sets of diameter $\leqslant 2t$.

For any set A of X, define $M_t(A)$ to be the number of members in subcover of β_t of minimal cardinality. Then as in [5], we define the lower metrical dimension $\underline{\dim}\, A$ of set A by

$$\underline{\dim}\, A = \underline{\lim}_{t\to 0} \frac{\log M_t(A)}{\log 1/t}$$

and define the dimension $\dim A$ of set A by

$$\dim A = \lim_{t\to 0} \frac{\log M_t(A)}{\log 1/t} \quad \text{if the limit}$$

exists.

Property 1 [5]. *Let X be an n-dimensional Euclidian space and suppose a compact subset A of X has interior points.*

*) As in [1], we write $\alpha \vee \beta = \{U \cap V : U \in \alpha, V \in \beta\}$ and we write $\alpha > \beta$ to mean that α is a refinement of β.

Then

$$\underline{\dim}\, A = \dim A = n.$$

Finally, when homeomorphism φ on a compact metric space has a positive real number $\lambda (1 \geqslant \lambda > 0)$ such that $d(\varphi(p), \varphi(q)) \geqslant \lambda \cdot d(p, q)$ for any $p, q \in M$, we call homeomorphism φ contractive and λ a contractive constant of φ.

§ 2. Lemmas and theorems.

Let M be a compact n-dimensional Riemannian manifold, $d: M \times M \to R$ a metric on M induced by some smooth Riemannian metric and let φ be a C^1-diffeomorphism on M. In this case we can obtain following lemma.

Lemma 1. *φ is contractive and a contractive constant is given by*

$$\lambda = \inf_{p \in M} \inf_{v_p \in T_p M} \frac{\|\varphi_* v_p\|}{\|v_p\|},$$

where $T_p M$ is tangent space at $p \in M$.

Proof. To prove Lemma 1, it is sufficient to consider the case of a connected manifold. Since φ is a C^1-diffeomorphism and $\{v_p \mid \|v_p\| = 1, v_p \in T_p M\}$ is a compact subset of $T_p M$, the smoothness of Riemannian metric implies that

$$\inf_{p \in M} \inf_{v_p \in T_p M} \frac{\|\varphi_* v_p\|}{\|v_p\|} = \inf_{p \in M} \inf_{\|v_p\| = 1} \|\varphi_* v_p\| = \lambda > 0.$$

By definition, the metric $d(p, q)$ is given by

$$d(p, q) = \inf L(c; a, b)$$

where $c: I = (a, b) \to M$ is a C^1-curve satisfying $c(a) = p$ and $c(b) = q$, and $L(c; a, b) = \int_a^b \|v_{c(t)}\| dt$. For $\varphi(p)$ and $\varphi(q)$, there exists a curve $\varphi \circ c'$, where c' is a curve joining p and q. From the definition of λ,

$$L(\varphi \circ c'; a', b') = \int_{a'}^{b'} \|\varphi_* v_{c'(t)}\| dt$$
$$\geqslant \lambda \int_{a'}^{b'} \|v_{c'(t)}\| dt \geqslant \lambda d(p, q).$$

In the next lemma we apply the elementary sublemma.

Sublemma. *Suppose $\{a^{(i)}(t)\}$, $i = 1, 2, \cdots, k$, are positive integer valued functions defined on $(0, \delta)$ such that*

$$\lim_{t \to 0} \frac{\log a^{(i)}(t)}{\log 1/t} = a^{(i)} \quad \text{exists for all } i.$$

Then

$$\lim_{t \to 0} \frac{\log \left(\sum_{i=1}^k a^{(i)}(t)\right)}{\log 1/t} = \max (a^{(1)}, a^{(2)}, \cdots, a^{(k)}).$$

Lemma 2. *Let M be a compact n-dimensional Riemannian manifold.*

Then

$$\dim M = n.$$

Proof. For any $p \in M$, there exists a convex chart (U, ψ) on M, that is two arbitrary points in U can be joint by a geodesic segment contained in U, where $\psi(U)$ is also convex. Without loss of generality, we can suppose that the diffeomorphism ψ is defined on $\bar{U}$. Let ρ be the usual metric on a n-dimensional Euclidean space. From the compactness of $\psi(\bar{U})$ and the convexity of $\bar{U}$ and $\psi(\bar{U})$, we can deduced that there exists a constant $1 \geqslant \mu > 0$ such that

$$\text{(c)} \qquad \frac{1}{\mu} d(p, q) \geqslant \rho(\psi(p), \psi(q)) \geqslant \mu d(p, q) \qquad \text{for all } p, q \in \bar{U}.$$

Proof of this is similar to that for Lemma 1. Now for any $t > 0$, let β_t be any cover of $\bar{U}$ (by arbitrary sets) with diam $\beta_t \leqslant 2t$. Then $\psi(\beta_t)$ is a cover of $\psi(\bar{U})$ with diam $\psi(\beta_t) \leqslant 2t/\mu$. Thus

$$M_t(\bar{U}) \geqslant M_{t/\mu}(\psi(\bar{U})), \qquad \text{and}$$

$$\underline{\lim}_{t\to 0} \frac{\log M_t(\bar{U})}{\log 1/t} \geqslant \underline{\lim}_{t\to 0} \frac{\log M_{t/\mu}(\psi(\bar{U}))}{\log \mu/t} \cdot \frac{\log \mu/t}{\log 1/t}.$$

Property 1 implies

$$\underline{\lim}_{t\to 0} \frac{\log M_t(\bar{U})}{\log 1/t} \geqslant n.$$

On the other hand, we can get similarly,

$$n \geqslant \overline{\lim}_{t\to 0} \frac{\log M_t(\bar{U})}{\log 1/t}.$$

Therefore

$$\dim \bar{U} = n.$$

For all $p \in M$, there exists such a convex chart (U_p, ψ_p). From the compactness of M, there exist finite convex charts $U_1, \cdots, U_k$ satisfying

$$\bigcup_{i=1}^{k} U_i = M.$$

Using a sublemma, we can show

$$\overline{\lim}_{t\to 0} \frac{\log M_t(M)}{\log 1/t} \leqslant \overline{\lim}_{t\to 0} \frac{(\log \sum_{i=1}^{k} M_t(\bar{U}_i))}{\log 1/t} = \lim_{t\to 0} \frac{\log (\sum_{i=1}^{k} M_t(\bar{U}_i))}{\log 1/t} = n.$$

On the other hand

$$n = \lim_{t\to 0} \frac{\log M_t(\bar{U}_i)}{\log 1/t} \leqslant \lim_{t\to 0} \frac{\log M_t(M)}{\log 1/t}.$$

Therefore we get $\dim M = n$.

Remark. Lemmas 1, 2 are also true in the case of a smooth compact Riemannian manifold with boundary. Proof of Lemma 2 is more complex in this case. Roughly speaking, when we consider the metric ρ to be induced from a curve on R^n, the relation (*c*) is true in Lemma 2. Moreover by compactness, there exists a positive large number M satisfying $M\rho'(\psi(p), \psi(q)) \geqslant \rho(\psi(p), \psi(q))$, where ρ' is a usual metric on R^n. Take a sufficiently small constant μ, then the relation (c) for a usual metric ρ' is also true.

Theorem 1. *Let M be an n-dimensional compact Riemannian manifold, φ a C^1-diffeomorphism on M.*

Then the topological entropy of φ is finite and satisfies

$$h(\varphi)\leqslant n\log 1/\lambda, \qquad 1\geqslant\lambda>0,$$

where λ is a contractive constant of φ. In particular, λ is chosen by $\lambda=\inf_{p\in M}\inf_{v_p\in T_pM}\frac{\|\varphi_* v_p\|}{\|v_p\|}$.

Proof. From Lemma 1, it follows that for any $t>0$

$$\alpha_t\vee\varphi\alpha_t\vee\cdots\vee\varphi^{m-1}\alpha_t<\beta_{\lambda^{m-1}t/3},$$

where α_t is the collection of all open spheres of radius t and $\beta_{\lambda^{m-1}t/3}$ is any cover of M with diam $\beta_{\lambda^{m-1}t/3}\leqslant 2\lambda^{m-1}t/3$. Therefore, $N(\alpha_t\vee\varphi\alpha_t\vee\cdots\vee\varphi^{m-1}\alpha_t)\leqslant M_{\lambda^{m-1}t/3}(M)$. From this, it follows that

$$\lim_{m\to\infty}\frac{1}{m}\log N(\alpha_t\vee\varphi\alpha_t\vee\cdots\vee\varphi^{m-1}\alpha_t)$$
$$\leqslant\lim_{m\to\infty}\frac{\log M_{\lambda^{m-1}t/3}(M)}{\log 3/\lambda^{m-1}t}\cdot\frac{\log 3/\lambda^{m-1}t}{m}.$$

Now, from Lemma 2,

$$\lim_{m\to\infty}\frac{\log M_{\lambda^{m-1}t/3}(M)}{\log 3/\lambda^{m-1}t}=n \quad\text{and,}\quad \lim_{m\to\infty}\frac{\log 3/\lambda^{m-1}t}{m}=\log 1/\lambda.$$

Thus, $h(\alpha_t,\varphi)\leqslant n\log 1/\lambda$ for all $t>0$. From the definition of $h(\varphi)$ it follows that

$$h(\varphi)\leqslant n\log 1/\lambda. \qquad \text{q.e.d.}$$

The idea of Theorem 1 above can be used to prove a more general result.

Theorem 2. *Let X be a compact metric space, and assume that a homeomorphism φ has a contractive constant $1\geqslant\lambda>0$.*

Then

$$h(\varphi)\leqslant\underline{\dim}\,(X)\log 1/\lambda.$$

Now the following theorem was proved by L. Goodwyn.

Theorem [3]. *Let X be a compact metric space, μ a probability measure on X and let φ be a homeomorphism on X preserving the measure μ.*

Then

$$h_\mu(\varphi)\leqslant h(\varphi),$$

where $h_\mu(\varphi)$ is the measure theoretic entropy [7].

From this theorem and Theorem 1, the following sharper form of the theorem of Kuchnirenko [6] and Avez [2] can be proved.

Theorem 3. *Let (M,μ,φ) be a classical dynamical system, that is to say, M is an n-dimensional compact Riemannian manifold, μ is a probability measure and the C^1-diffeomorphism φ is measure preserving.*

Then

$$h_\mu(\varphi) \leqslant n \log 1/\lambda,$$

where λ *is a contractive constant of* φ. *In particular,* λ *is chosen by*

$$\lambda = \inf_{p \in M} \inf_{v_p \in T_p M} \frac{\|\varphi_* v_p\|}{\|v_p\|}.$$

§ 3. Examples on a flow.

Let $\{\varphi_t \mid -\infty < t < \infty\}$ be a flow, that is a one parameter group of diffeomorphisms on M. In [4], we could consider a topological entropy of a flow. Thus we can derive the following estimate.

Theorem 4. *Let* M *be an* n*-dimensional compact Riemannian manifold and let* $\{\varphi_t\}$ *be a flow on* M.

Then the topological entropy of $\{\varphi_t\}$ *satisfies*

$$h(\varphi_1) = \frac{1}{|t|} h(\varphi_t) \leqslant \frac{1}{|t|} n \log 1/\lambda(t) \qquad (t \neq 0),$$

where

$$\lambda(t) = \inf_{p \in M} \inf_{v_p \in T_p M} \frac{\|(\varphi_t)_* v_p\|}{\|v_p\|}.$$

Example 1. Let M be a compact connected n-dimensional Riemann manifold. If the Gaussian curvature R is non negative, then the geodesic flow on the unitary tangent bundle T_1M has a zero topological entropy.

Proof. Use Theorem 4 and observe

$\lambda(1/\sqrt{R}) = 1$ as $R > 0$ and $1/\lambda(t) \leqslant 1 + t$ as $R = 0$.

Example 2. A holocycle flow has a zero topological entropy.

Proof. Use Theorem 4 and observe that $1/\lambda(t)$ is bounded from above by a polynomial $P(t)$.

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