47. *On Homogeneous Complex Manifolds with Negative Definite Canonical Hermitian Form*

By Hirohiko SHIMA
Osaka University

(Comm. by Kenjiro SHODA, M. J. A., March 12, 1970)

Throughout this note, G denotes a connected Lie group and K is a closed subgroup of G. We assume that G acts effectively on the homogeneous space G/K. Suppose that G/K carries a G-invariant complex structure I and a G-invariant volume element v. Then we may define canonical hermitian form associated to I and v [2].

Theorem. *Let G/K be a homogeneous complex manifold with a G-invariant volume element. If the canonical hermitian form h of G/K is negative definite, then G is a semisimple Lie group.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of all left invariant vector fields on G and $\mathfrak{k}$ the subalgebra of $\mathfrak{g}$ corresponding to K. We denote by I the G-invariant complex structure tensor on G/K. Let π_e be the differential of the canonical projection π from G onto G/K at the identity e and let $I_{e'}$ (resp. X_e) be the value of I (resp. $X \in \mathfrak{g}$) at $\pi(e)=e'$ (resp. e). Koszul [2] proved that there exists a linear endomorphism J of $\mathfrak{g}$ such that for $X, Y \in \mathfrak{g}$ and $W \in \mathfrak{k}$

$$\pi_e(JX)_e = I_{e'}(\pi_e X_e) \tag{1}$$

$$J\mathfrak{k} \subset \mathfrak{k} \tag{2}$$

$$J^2X \equiv -X \bmod \mathfrak{k} \tag{3}$$

$$[JX, W] \equiv J[X, W] \bmod \mathfrak{k} \tag{4}$$

$$[JX, JY] \equiv J[JX, Y] + J[X, JY] + [X, Y] \bmod \mathfrak{k} \tag{5}$$

Moreover, the canonical hermitian form h of G/K associated to the G-invariant volume element is expressed as follows. Putting

$$\eta = \pi^* h,$$

$$\eta(X, Y) = \frac{1}{2}\psi([JX, Y]) \tag{6}$$

for $X, Y \in \mathfrak{g}$, where $\psi(X) = \text{trace of } (ad(JX) - Jad(X))$ on $\mathfrak{g}/\mathfrak{k}$ for $X \in \mathfrak{g}$. As h is assumed to be negative definite, $\eta(X, X) \leqq 0$ for any $X \in \mathfrak{g}$, and $\eta(X, X) = 0$ if and only if $X \in \mathfrak{k}$. Therefore, putting $\omega = -\psi$, $(\mathfrak{g}, \mathfrak{k}, J, \omega)$ is a j-algebra in the sense of E. B. Vinberg, S. G. Gindikin and I. I. Pjateckii-Šapiro [4].

Now suppose that $\mathfrak{g}$ is not a semisimple Lie algebra. Then there is a non-zero commutative ideal $\mathfrak{r}$ of $\mathfrak{g}$. Consider the J-invariant subalgebra

$$\mathfrak{g}'=\mathfrak{k}+J\mathfrak{r}+\mathfrak{r}$$

It is known [4] that there exists an affine homogeneous convex domain U in $\mathfrak{r}$, not containing any straight line, such that the j-algebra $(\mathfrak{g}', \mathfrak{k}, J, \omega)$ is isomorphic to the j-algebra of the tube domain $\mathscr{D}(U) =\{X+\sqrt{-1}\,Y : X \in \mathfrak{r}, Y \in U\}$. More precisely, if G' (resp. K^0) denotes the connected Lie subgroup of G corresponding to $\mathfrak{g}'$ (resp. $\mathfrak{k}$), the complex structure I of G/K induces a G'-invariant complex structure of G'/K^0 and G'/K^0 is locally holomorphically equivalent to $\mathscr{D}(U)$. Since $\mathscr{D}(U)$ is holomorphically equivalent to a bounded domain, the canonical hermitian form h' of G'/K^0 is positive definite. Now, again by [2], h' is expressed as follows. Putting $\eta'=\pi^* h'$,

$$\eta'(X', Y')=\frac{1}{2}\psi'([JX', Y'])$$

for $X', Y' \in \mathfrak{g}'$, where $\psi'(X')=$trace of $(ad(JX')-Jad(X'))$ on $\mathfrak{g}'/\mathfrak{k}$ for $X' \in \mathfrak{g}'$. On the other hand, by [4], there exists a unique non-zero element $E \in \mathfrak{r}$, such that for $X \in \mathfrak{r}$

$$\omega(X)=\omega([JE, X]) \tag{7}$$

$$[JE, E]=E \tag{8}$$

$$[JE, \mathfrak{k}]\subset \mathfrak{k} \tag{9}$$

$$[E, \mathfrak{k}]=\{0\} \tag{10}$$

Using (4), (5), (10) and the fact that $\mathfrak{r}$ is a commutative ideal,

$$ad(JE)(W+JX+Y)\equiv J[JE, X]+J[E, JX]+[JE, Y] \bmod \mathfrak{k}$$

for $X, Y \in \mathfrak{r}$ and $W \in \mathfrak{k}$. Therefore we obtain

$$ad(JE)\mathfrak{g}'\subset \mathfrak{g}'$$

As $\mathfrak{r}$ is an ideal of $\mathfrak{g}$,

$$Jad(E)\mathfrak{g}\subset J\mathfrak{r}$$

Hence it follows that

$$\begin{aligned} 2\eta(E, E) &= \psi([JE, E]) \\ &= \psi(E) \\ &= \text{trace of } (ad(JE)-Jad(E)) \text{ on } \mathfrak{g}/\mathfrak{k} \\ &= \text{trace of } (ad(JE)-Jad(E)) \text{ on } \mathfrak{g}'/\mathfrak{k} \\ &\quad + \text{trace of } (ad(JE)-Jad(E)) \text{ on } \mathfrak{g}/\mathfrak{g}' \\ &= \psi'(E)+\text{trace of } ad(JE) \text{ on } \mathfrak{g}/\mathfrak{g}' \\ &= 2\eta'(E, E)+\text{trace of } ad(JE) \text{ on } \mathfrak{g}/\mathfrak{g}' \end{aligned}$$

As h' is positive definite, $\eta'(E, E)>0$. By [4], the real parts of the eigenvalues of $ad(JE)$ on $\mathfrak{g}/\mathfrak{g}'$ are equal to 0 or 1/2, so the trace of $ad(JE)$ on $\mathfrak{g}/\mathfrak{g}'\geqq 0$. These imply that $2\eta'(E, E)+$trace of $ad(JE)$ on $\mathfrak{g}/\mathfrak{g}'>0$. On the other hand, as h is negative definite, $\eta(E, E)<0$, which is a contradiction. Hence $\mathfrak{g}$ must be semisimple. q.e.d.

Under the assumption of the Theorem, $-h$ defines a G-invariant Kähler structure on G/K, hence K is compact and equal to the centralizer of a one parameter subgroup of G, and G must be compact [2].

Conversely, if G is a compact semisimple Lie group and if G/K carries a G-invariant Kähler structure, then the canonical hermitian form of G/K is negative definite [2]. We know also that the canonical hermitian form of a homogeneous Kähler manifold is equal to the Ricci curvature. Therefore we have the following

Corollary. *Let G/K be a homogeneous Kähler manifold of a connected Lie group G. The Ricci curvature of G/K is negative definite if and only if G is a compact semisimple Lie group.*

Remark. Hano [1] proved that if G is unimodular and if the Ricci curvature of a homogeneous Kähler manifold G/K is non-degenerate, then G is a semisimple Lie Group.

References

[1] J. Hano: On kaehlerian homogeneous spaces of unimodular Lie groups. Amer. J. Math., **79**, 885–900 (1957).

[2] J. L. Koszul: Sur la forme hermitienne canonique des espaces homogènes complexes. Canad. J. Math., **7**, 562–576 (1955).

[3] ——: Domaines bornés homogènes et orbites de groupes de transformations affines. Bull. Soc. Math. France, **89**, 515–533 (1961).

[4] E. B. Vinberg, S. G. Gindikin, and I. I. Pjateckii Šapiro: Classification and canonical realization of complex bounded homogeneous domains. Trudy Moscow Math. Obshch., **12**, 359–388 (1963); Trans. Moscow Math. Soc., **12**, 404–437 (1963).