81. *Notes on Modules. II*

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Generalizing a well known important result (cf. Jacobson [1], Chapter IV, p. 93) for vector spaces, in our paper all twosided ideals of the total endomorphism ring $E(M)$ of a homogeneous completely reducible module M over an arbitrary ring A are determined. Our result is an English version of the earlier paper of the author [2].

Theorem. *Let $E(M)$ be the total endomorphism ring of a homogeneous completely reducible right A-module M over an arbitrary ring A. Then for every nonzero twosided ideal $\mathcal{J}$ of $E(M)$ there exists an infinite cardinality $\mathfrak{M}$ such that $\mathcal{J}$ coincides with the set of all endomorphisms γ of M with* rang $\gamma M<\mathfrak{M}$

Proof. We assume that rang $M \geqq \aleph_0$ over A, being $E(M)$ a simple total matrix ring over a division ring for the particular case

$$\text{rang } M<\aleph_0.$$

1. Firstly we assert that if $\mathcal{J}$ is a twosided ideal of $E(M)$ with $\gamma_2 \in \mathcal{J}$ and

$$\text{rang } \gamma_1 M \leqq \text{rang } \gamma_2 M \tag{1}$$

for an arbitrary $\gamma_1 \in E(M)$, then $\gamma_1 \in \mathcal{J}$

Namely, for $i=1$ and $i=2$ let N_i be the kernel of the endomorphism γ_i in M. Then there exists a completely reducible submodule K_i of M with $M=N_i \oplus K_i$. Then (1) implies

$$\text{rang } K_1 \leqq \text{rang } K_2 \tag{2}$$

If $K_i=\sum \oplus \{\underset{(i)}{k_{\alpha_j}}\}$, then by (2) and by the fact that M is homogeneous, there exists an endomorphism $\delta_1 \in E(M)$ such that holds

$$\delta_1 \underset{(1)}{k_{\alpha_1}}=\underset{(2)}{k_{\alpha_1'}} \quad \text{and} \quad \delta_1 N_1=0 \tag{3}$$

Here α_1' denotes an uniquely determined index α_2 from Γ_2, and for $\alpha_1 \neq \beta_1$ one has obviously $\alpha_1' \neq \beta_1'$ ($\alpha_1, \beta_1 \in \Gamma_1$; $\alpha_2, \beta_2 \in \Gamma_2$, being Γ_2 the set of indices of fixed basis elements of K_i). Consequently, the restriction of δ_1 on $\delta_1 K_1$ has an inverse element δ_1^{-1}.

From an assumed linear connection

$$\sum_{j=1}^{n} \gamma_2 \delta_1 \underset{(1)}{k_{\alpha_j}} a_j=0 \quad (a_j \in A) \tag{4}$$

follows $\gamma_2 k^*=0$ for the element

$$k^*=\sum_{j=1}^{n} \delta_1 \underset{(1)}{k_{\alpha_j}} a_j \in K_2$$

Therefore $k^* \in N_2 \cap K_2$, and $k^*=0$. There exists an inverse element

δ_1^{-1} of the restriction of δ_1 on $\delta_1 K_1$, so one has

(5) $$\delta_1^{-1}k^* = \sum_{j=1(1)}^{n} k_{\alpha_j} a_j = 0$$

which yields $\underset{(1)}{k_{\alpha_j}} a_j = 0$ for every $j=1, 2, \cdots, n$, forming $\sum\{k_\alpha\}$ a direct sum. Therefore, the elements $\gamma_2 \delta_1 \underset{(1)}{k_\alpha}$ are linearly independent over A $(\alpha \in \Gamma_1)$. By the fact that M is homogeneous, there exists an element $\delta_2 \in E(M)$ satisfying

(6) $$\delta_2(\gamma \delta_1 \underset{(1)}{k_\alpha}) = \gamma_1 \underset{(1)}{k_\alpha}.$$

Analysing the difference $\gamma_0 = \delta_2 \gamma_2 \delta_1 - \gamma_1$, we conclude, $\gamma_0 = 0$, that is

(7) $$\gamma_1 = \delta_2 \gamma_2 \delta_1 \in \mathcal{J},$$

which completes the proof of Assertion 1.

2. Secondly, it can be shown that for rang $M \geqq \aleph_0$ and for every nonzero twosided ideal $\mathcal{J}$ of $E(M)$, the endomorphisms γ with condition rang $\gamma M < \aleph_0$ are contained in $\mathcal{J}$, and all these endomorphisms γ form a twosided ideal F of $E(M)$.

Namely, for the direct composition $M = \Sigma \oplus \{m_\alpha\} (\alpha \in \Gamma)$ we define the endomorphisms $\varepsilon_\beta \in E(M)$ by

$$\varepsilon_\beta m_\alpha = \delta_{\alpha\beta} m_\beta.$$

(8) $$\varepsilon_\beta m_\alpha a = \delta_{\alpha\beta} m_\alpha a (\alpha_1 \beta \in \Gamma, a \in A)$$

where $\delta_{\alpha\beta}$ denotes Kronecker's delta symbol. Clearly rang $\varepsilon_\alpha M = 1$ and thus by Assertion 1, holds $\varepsilon_\beta \in \mathcal{J}$ for every β. Consequently

(9) $$\delta_{\beta_1} + \varepsilon_{\beta_2} + \cdots + \varepsilon_{\beta_3} \in \mathcal{J}$$

which verifies the existence of endomorphisms $\gamma \in \mathcal{J}$ with rang $\gamma M = n$ for every n.

From this follows already every statement of Assertion 2.

3. Thirdly, we prove that there exists for every nonzero ideal $\mathcal{J}$ of $E(M)$ an infinite cardinality $\mathfrak{M}$, such that $\mathcal{J}$ consists of every endomorphism $\gamma \in E(M)$ satisfying rang $\gamma M < \mathfrak{M}$

Let $\mathfrak{M}$ be namely the least (infinite) cardinality satisfying rang $\gamma M < \mathfrak{M}$ for every $\gamma \in \mathcal{J}$. Clearly there exists such a cardinality. By Assertion 2, one has $F \subseteq \mathcal{J}$ and thus $\mathfrak{M} \geqq \aleph_0$.

If rang $M < \mathfrak{M}$, then by definition of $\mathfrak{M}$ there exists an element $\gamma \in \mathcal{J}$ with the condition rang $\gamma M =$ rang M and by Assertion 1 also $\mathcal{J} = E(M)$.

Assuming that $\mathcal{J} \neq E(M)$ and $\mathcal{J} \neq 0$, in case $\mathfrak{M} = \aleph_0$ one has $\mathcal{J} = F$ by Assertion 2.

Furthermore, in case $\mathfrak{M} > \aleph_0$ and $\mathfrak{M} \leqq$ rang M the condition rang $\eta M < \mathfrak{M}$ and definition of $\mathfrak{M}$ imply the existence of an endomorphism $\vartheta \in \mathcal{J}$, with

(10) $$\text{rang } \vartheta M \geqq \text{rang } \eta M$$

whence by Assertion 1 follows $\eta \in \mathcal{J}$.

These Assertions 1, 2 and 3 complete the proof of the Theorem.

References

[1] N. Jacobson: Structure of Rings. Providence (1964).
[2] F. Szász: A teljesen reducibilis operátormodulusokról (On the completely reducible operator modules). Magyar Tudományos Akadémia. III. Osztályának Közleményei, **11** (4), 417–425 (1961).