77. On Nest Algebras of Operators

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1. In this paper we study certain algebras of operators termed 'nest algebras', which were introduced by J. R. Ringrose [3]. Our main results (Theorems 4 and 5) are concerned with characterizations of such algebras, and consequently it is proved that each weakly closed maximal triangular operator algebra is hyperreducible.

Throughout this paper the terms *Hilbert space*, *subspace*, *operator*, *projection* are used to mean *complex* Hilbert space, *closed linear* subspace, *bounded linear* operator, *orthogonal* projection, respectively. Given a subspace $\mathfrak{M}$ of a Hilbert space $\mathfrak{H}$, we shall write $P_{\mathfrak{M}}$ for the projection from $\mathfrak{H}$ onto $\mathfrak{M}$, and $\mathfrak{H}\ominus\mathfrak{M}$ for the orthogonal complement of $\mathfrak{M}$ in $\mathfrak{H}$. If $\{\mathfrak{M}_\alpha\}$ is a collection of subspaces of $\mathfrak{H}$, then the smallest subspace which contains each $\mathfrak{M}_\alpha$ will be denoted by $\vee\mathfrak{M}_\alpha$, and the largest subspace contained in each $\mathfrak{M}_\alpha$ will be denoted by $\wedge\mathfrak{M}_\alpha(=\cap\mathfrak{M}_\alpha)$. Set inclusion in the wide sense will be denoted by the symbol '$\subseteq$', and we reserve '$\subset$' for proper inclusion.

The class of all operators from a Hilbert space $\mathfrak{H}$ into itself will be denoted by $\mathcal{L}(\mathfrak{H})$. By an *algebra of operators* on $\mathfrak{H}$ we shall mean a subset $\mathcal{A}$ of $\mathcal{L}(\mathfrak{H})$ such that, if λ is a complex number and $A, B\in\mathcal{A}$, then $A+B, AB, \lambda A\in\mathcal{A}$. A self-adjoint algebra of operators will be termed a $*$-algebra.

2. Following J. R. Ringrose, a family $\mathcal{N}$ of subspaces of a Hilbert space $\mathfrak{H}$ will be called a *nest* if it is totally ordered by the inclusion relation $\subseteq$; $\mathcal{N}$ will be called a *complete* nest if, further,

(i) $(0), \mathfrak{H}\in\mathcal{N}$;

(ii) given any subnest $\mathcal{N}_0$ of $\mathcal{N}$, the subspaces $\bigwedge_{\mathfrak{M}\in\mathcal{N}_0}\mathfrak{M}$, $\bigvee_{\mathfrak{M}\in\mathcal{N}_0}\mathfrak{M}$ are both members of $\mathcal{N}$.

Given a complete nest $\mathcal{N}$ and a non-zero subspace $\mathfrak{M}$ in $\mathcal{N}$, we define

$$\mathfrak{M}_-=\vee\{\mathfrak{N}\,|\,\mathfrak{N}\in\mathcal{N}, \mathfrak{N}\subset\mathfrak{M}\}.$$

Clearly $\mathfrak{M}_-\in\mathcal{N}$.

If $\mathcal{N}$ is a complete nest of subspaces of a Hilbert space $\mathfrak{H}$, then the *nest algebra* $\mathcal{A}_{\mathcal{N}}$ associated with $\mathcal{N}$ is defined to be the class of all operators on $\mathfrak{H}$ which leave invariant each subspace in $\mathcal{N}$. Clearly $\mathcal{A}_{\mathcal{N}}$ is a weakly closed subalgebra of $\mathcal{L}(\mathfrak{H})$.

The following lemma, included here for the sake of completeness, is restatement of ([3], Theorem 3.4).

Lemma 1. *Let $\mathfrak{N}$ be a complete nest of subspaces of a Hilbert space $\mathfrak{H}$, and let $\mathfrak{M}$ be a subspace of $\mathfrak{H}$ which is invariant under each operator in $\mathcal{A}_{\mathfrak{N}}$. Then $\mathfrak{M} \in \mathfrak{N}$.*

Let $\mathcal{R}$ be a subalgebra of $\mathcal{L}(\mathfrak{H})$, and let $\mathcal{R}^* = \{A^* \mid A \in \mathcal{R}\}$. Then after terminologies for the triangular algebras of operators in [1], the sub $*$-algebra $\mathfrak{A} = \mathcal{R} \cap \mathcal{R}^*$ will be called the *diagonal* of $\mathcal{R}$, and the *core* of $\mathcal{R}$ is defined to be the von Neumann algebra generated by the projections onto subspaces of $\mathfrak{H}$ which are invariant under $\mathcal{R}$.

Lemma 2. *Let $\mathfrak{A}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$, and let $\mathcal{R}$ be an algebra of operators such that its diagonal is $\mathfrak{A}$. Let $\mathfrak{M}$ be a subspace of $\mathfrak{H}$ which is invariant under $\mathcal{R}$. Then $P_{\mathfrak{M}} \in \mathfrak{A}'$.*

Proof. Suppose that A is a self-adjoint element of $\mathfrak{A}$. Then we have

$$AP_{\mathfrak{M}} = P_{\mathfrak{M}}AP_{\mathfrak{M}} = P_{\mathfrak{M}}A^*P_{\mathfrak{M}} = (P_{\mathfrak{M}}AP_{\mathfrak{M}})^* = (AP_{\mathfrak{M}})^* = P_{\mathfrak{M}}A.$$

Since $*$-algebra $\mathfrak{A}$ is generated by the self-adjoint elements in itself, we have $P_{\mathfrak{M}} \in \mathfrak{A}'$.

Lemma 3. *Let $\mathfrak{N}$ be a complete nest of subspaces of a Hilbert space $\mathfrak{H}$, and let $\mathfrak{A}$ be the diagonal of nest algebra $\mathcal{A}_{\mathfrak{N}}$ associated with $\mathfrak{N}$. Then the core of $\mathcal{A}_{\mathfrak{N}}$ is $\mathfrak{A}'$ (the commutant of $\mathfrak{A}$).*

Proof. Let $\mathcal{B}$ be the core of $\mathcal{A}_{\mathfrak{N}}$. Then by definition and Lemma 1, $\mathcal{B}$ is the von Neumann algebra generated by the set of projections $\{P_{\mathfrak{M}} \mid \mathfrak{M} \in \mathfrak{N}\}$. And by Lemma 2 $P_{\mathfrak{M}} \in \mathfrak{A}'$ for each $\mathfrak{M} \in \mathfrak{N}$. Hence we have

$$\mathcal{B} \subseteq \mathfrak{A}'.$$

If $B \in \mathcal{B}'$ and $B^* = B$, then B commutes with $\mathcal{B}$ and, all the more, with each $P_{\mathfrak{M}} (\mathfrak{M} \in \mathfrak{N})$. Hence $B \in \mathcal{A}_{\mathfrak{N}}$ and $B \in \mathcal{A}_{\mathfrak{N}} \cap \mathcal{A}_{\mathfrak{N}}^* = \mathfrak{A}$. Since $\mathcal{B}'$ is generated by the self-adjoint operators in itself, we have $\mathcal{B}' \subseteq \mathfrak{A}$ and then

$$\mathcal{B} \supseteq \mathfrak{A}'.$$

This completes the proof of the lemma.

The following result is a characterization of nest algebras in a sense.

Theorem 4. *Let $\mathfrak{A}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$ such that its commutant $\mathfrak{A}'$ is abelian, and let $\mathcal{R}$ be an algebra of operators such that its diagonal is $\mathfrak{A}$, i.e., $\mathcal{R} \cap \mathcal{R}^* = \mathfrak{A}$. Then $\mathcal{R}$ is a nest algebra if and only if the following conditions are satisfied:*

1° *$\mathcal{R}$ is maximal with respect to the property of having $\mathfrak{A}$ as its diagonal;*

2° *$\mathfrak{A}'$ is the core of $\mathcal{R}$.*

Proof. Suppose that $\mathcal{R}$ satisfies Conditions 1° and 2° and that $\mathfrak{M}$ is any subspace of $\mathfrak{H}$ which is invariant under $\mathcal{R}$. Then by Lemma

2 we have clearly $P_{\mathfrak{M}} \in \mathfrak{A}' \subseteq \mathfrak{A}'' = \mathfrak{A} \subseteq \mathfrak{R}$. Therefore by virtue of ([1], Lemmas 2.3.2 and 2.3.3) and our Condition 1°, the family $\mathcal{N}$ of the invariant subspaces under $\mathfrak{R}$ is a complete nest. Let $\mathcal{A}_{\mathcal{N}}$ be the nest algebra associated with $\mathcal{N}$. Then clearly $\mathfrak{R} \subseteq \mathcal{A}_{\mathcal{N}}$. Suppose that $A \in \mathcal{A}_{\mathcal{N}} \cap \mathcal{A}_{\mathcal{N}}^*$ and $A^* = A$. Then we have for each $\mathfrak{M} \in \mathcal{N}$,

$$AP_{\mathfrak{M}} = P_{\mathfrak{M}} A P_{\mathfrak{M}} = P_{\mathfrak{M}} A^* P_{\mathfrak{M}} = (P_{\mathfrak{M}} A P_{\mathfrak{M}})^* = (AP_{\mathfrak{M}})^* = P_{\mathfrak{M}} A.$$

Hence by Condition 2°, $A \in \mathfrak{A}'' = \mathfrak{A}$. Since a $*$-algebra is generated by the self-adjoint elements in itself, we have

$$\mathcal{A}_{\mathcal{N}} \cap \mathcal{A}_{\mathcal{N}}^* \subseteq \mathfrak{A}.$$

On the other hand, it follows from the above mentioned property $\mathfrak{R} \subseteq \mathcal{A}_{\mathcal{N}}$ that we have

$$\mathcal{A}_{\mathcal{N}} \cap \mathcal{A}_{\mathcal{N}}^* \supseteq \mathfrak{A}.$$

Hence $\mathcal{A}_{\mathcal{N}} \cap \mathcal{A}_{\mathcal{N}}^* = \mathfrak{A}$, and by virtue of Condition 1°, $\mathfrak{R} = \mathcal{A}_{\mathcal{N}}$.

Suppose conversely that $\mathfrak{R}$ is the nest algebra associated with a complete nest $\mathcal{N}$. Then by Lemma 3, $\mathfrak{R}$ satisfies Condition 2°. Therefore it remains to prove that $\mathfrak{R}$ satisfies also Condition 1°. But the very same argument as one used by Kadison and Singer in proof of ([1], Theorem 3.1.1) gives the following result: Let $\mathcal{S}$ be an algebra which contains $\mathfrak{R}$ and is maximal with respect to the property of having $\mathfrak{A}$ as its diagonal. Then we have for each $S \in \mathcal{S}$ and each $\mathfrak{M} \in \mathcal{N}$,

$$(I - P_{\mathfrak{M}}) S P_{\mathfrak{M}} = 0.$$

We omit repeating of the argument. Hence $S \in \mathfrak{R}$ and then $\mathcal{S} = \mathfrak{R}$. This completes the proof of the theorem.

In the next we give another characterization of nest algebras.

Theorem 5. *Let $\mathfrak{A}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$ such that its commutant $\mathfrak{A}'$ is abelian, and let $\mathfrak{R}$ be a weakly closed subalgebra of $\mathcal{L}(\mathfrak{H})$ which is maximal with respect to the property of having $\mathfrak{A}$ as its diagonal. Then $\mathfrak{R}$ is a nest algebra.*

Proof. Let $\mathcal{N}$ be the family of the subspaces of $\mathfrak{H}$ which are invariant under $\mathfrak{R}$. Then we proved, in proof of Theorem 4, that $\mathcal{N}$ is a complete nest. Hence by virtue of ([2], Theorem 2) it will suffice to show that $\mathfrak{R}$ contains a maximal abelian self-adjoint algebra. Since $\mathfrak{A}' \subseteq \mathfrak{A}$, there is a maximal abelian sub $*$-algebra $\mathcal{B}$ of $\mathfrak{A}$ which contains $\mathfrak{A}'$, by Zorn's lemma. Let X be any operator in $\mathcal{L}(\mathfrak{H})$ which commutes with each member B in $\mathcal{B}$. Then clearly X commutes with each member in $\mathfrak{A}'$. Hence $X \in \mathfrak{A}'' = \mathfrak{A}$. By maximality in $\mathfrak{A}$, $X \in \mathcal{B}$. Then $\mathcal{B}$ is a maximal abelian self-adjoint algebra in $\mathcal{L}(\mathfrak{H})$. This completes the proof of the theorem.

Corollary. *Each weakly closed maximal triangular algebra is hyperreducible.*

Remark. In the preparation of this paper we were informed by

'Contents of Contemporary Mathematical Journals, Vol. 1, No. 25, Dec. 12, (1969)' appearance of P. Rosenthal's paper which is entitled "*Weakly closed maximal triangular algebras are hyperreducible* (Proc. Amer. Math. Soc., Vol. 24, No. 1, Jan. 1970)" and does not still come to hand.

References

[1] R. V. Kadison and I. M. Singer: Triangular operator algebras. Amer. J. Math., **82**, 227–259 (1960).

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[3] J. R. Ringrose: On some algebras of operators. Proc. London Math. Soc., **15**, 61–83 (1965).