99. On the Sets of Points in the Ranked Space. IV

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In this paper we report some properties holding in a sequentially compact ranked space.

We defined in previous papers the concepts such as r-open subsets [1], sequentially compact subsets [2], R-convergence and para-convergence of a sequence of points [3], $\{p_\alpha\}$, in a ranked space.

In the present paper we define a concept that a ranked space R is countably compact and a concept that R is totally bounded.

Definition 1. A ranked space is countably compact if and only if every countable open covering of the ranked space, S, has a finite subcovering of S.

Definition 2. A ranked space R is totally bounded if and only if for every natural number γ, there are suitable finite points of R, $a_1, a_2, \cdots, a_n$, and $V(a_i)$ $(i=1,2,\cdots,n)$ such that

(1) $V(a_i)\in\mathfrak{V}_\gamma$ and $V(a_i)\cap V(a_j)=\emptyset$ $(i\neq j)$

and

(2) there does not exist any p of R and $V(p)\in\mathfrak{V}_\gamma$ satisfying

$$V(p)\subset R-\bigcup_{i=1}^{n}V(a_i).$$

Proposition 1. *If a ranked space R is sequentially compact, then R is countably compact.*

Proof. Suppose that R is not countably compact. Then, there is some countable open covering $\{U_i\}$ such that $\bigcup_{i=1}^{n}U_i\neq R$ for every natural number n. Hence there is a point a_n of R such that $a_n\in R-(\bigcup_{i=1}^{n}U_i)$ for every n. Since R is sequentially compact, the sequence $\{a_n\}$ has its subsequence $\{a_{m_\alpha}\}$ such that $a\in\{\lim_\alpha a_{m_\alpha}\}$. By the definition of R-convergence, there is a fundamental sequence $\{V_k(a)\}$ of neighborhoods of the point a for which there holds $a_{m_k}\in V_k(a)$.

On the other hand, since $a\in R$ and $\{U_i\}$ is a covering of R, there is an element U_l of $\{U_i\}$ such that $a\in U_l$. Since U_l is an r-open subset

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of R, there is an element $V_p(a)$ of $\{V_k(a)\}$ such that $V_p(a) \subset U_l$. Then we have $a_{m_p} \in V_p(a)$. Moreover, it is possible to choose a natural number p such that $m_p > l$. Then, by the construction of the sequence $\{a_n\}$, the element a_{m_p} of $\{a_m\}$ does not belong to U_l. This is a contradiction.

Proposition 2. *If a ranked space R is sequentially compact and satisfies the following condition, then R is totally bounded.*

Condition. If $a \in \{\lim a_n\}$, then for every rank γ_α and for every pair of a_m and $a_n \in V_\alpha(a) \in \mathfrak{V}_{\gamma_\alpha}$ such that $m, n > \alpha$, there are $V(a_m) \in \mathfrak{V}_{\gamma_\alpha}$ and $V(a_n) \in \mathfrak{V}_{\gamma_\alpha}$ such that $V(a_m) \cap V(a_n) \neq \emptyset$.

Proof. Suppose that R is not totally bounded. We can choose a sequence $\{a_i\}$ in R such that for some rank γ_α there hold $V(a_i) \in \mathfrak{V}_{\gamma_\alpha}$ for $i=1, 2, \cdots, V(a_i) \cap V(a_j) = \emptyset$ for $i \neq j$, and $R \supset \bigcup_{i=1}^{\infty} V(a_i)$. Since R is sequentially compact, for a suitable subsequence $\{a_{k_i}\}$ of $\{a_i\}$ there is an element a of R such that $a \in \{\lim_i a_{k_i}\}$. Now, suppose that $\alpha < k_n, k_m$, $k_n \neq k_m$ for $V_\alpha(a) \in \mathfrak{V}_{\gamma_\alpha}$, we have $a_{k_n}, a_{k_m} \in V_\alpha(a)$. Then by Condition, there are neighborhoods $V(a_{k_m})$ and $V(a_{k_n})$ with the rank γ_α such that $V(a_{k_m}) \cap V(a_{k_n}) \neq \emptyset$. This is a contradiction.

So far we have used R-convergence in order to define a sequentially compact set. Hence we replace R-convergence by para-convergence, so we have the following proposition.

Proposition 3. *If a ranked space R is sequentially compact and satisfies the following condition, then R is totally bounded.*

Condition. In R, for every $a \in R$ and arbitrary ranks γ_i, γ_j, if $\gamma_i \leq \gamma_j$, $V(a) \in \mathfrak{V}_{\gamma_i}$ and $U(a) \in \mathfrak{V}_{\gamma_j}$ then $V(a) \supseteq U(a)$.

Proof. Suppose that R is not totally bounded. We can choose a sequence $\{a_i\}$ in R such that for some rank γ_α there hold $V(a_i) \in \mathfrak{V}_{\gamma_\alpha}$ for $i=1, 2, \cdots$, $V(a_i) \cap V(a_j) = \emptyset$ for $i \neq j$, and $R \supset \bigcup_{i=1}^{\infty} V(a_i)$. Since R is sequentially compact, for a suitable subsequence $\{a_{k_i}\}$ of $\{a_i\}$ there is an element a of R such that $a \in \{\text{para} \lim_i a_{k_i}\}$. Hence, by Condition of para-convergence, there is a fundamental sequence $\{V_{k_i}(a_{k_i})\}$ such that $a \in V_{k_i}(a_{k_i}) \in \mathfrak{V}_{\gamma_{k_i}}$. Now suppose that $\alpha < k_i, k_j, k_i \neq k_j$ then we have $\gamma_\alpha < \gamma_{k_i}, \gamma_{k_j}$. Then by Condition, we have $V(a_{k_i}) \supset V_{k_i}(a_{k_i})$ and $V(a_{k_j}) \supset V_{k_j}(a_{k_j})$. Moreover, since $a \in V_{k_i}(a_{k_i}) \in \mathfrak{V}_{\gamma_{k_i}}$ and $a \in V_{k_j}(a_{k_j}) \in \mathfrak{V}_{\gamma_{k_j}}$, we have $V(a_{k_i}) \cap V(a_{k_j}) \neq \emptyset$. This is a contradiction.

References

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