98. *A Convergence Theorem in Measurable Function Spaces of Concave Type*[*)]

By Shozo KOSHI
Mathematical Institute, Okayama University

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1. L. Schwarz has shown that in $L^p(\Omega\cdot\mu)$ $(0\leqq p<+\infty)$ for every C-sequence its sum is convergent [3]. In this note, we shall show that this fact is true in some type of measurable function spaces. Let L be a measurable function space (topological vector space) with a linear topology $\mathcal{T}$. A sequence $f_n\in L(n=1,2,\cdots)$ is called *C-sequence* in L if $\sum_{n=1}^{\infty}c_nf_n$ converges with respect to $\mathcal{T}$ for all sequences of real numbers $\{c_n\}$ which tend to 0.

Now, we shall consider some class of function spaces which includes $L^p(0\leqq p\leqq 1)$.

2. Let Ω be a measure space with measure μ where Ω is a union of mutually disjoint measurable set $A_\lambda(\lambda\in\Lambda)$ with finite measure and every measurable set of finite measure is contained in at most countable union from $A_\lambda(\lambda\in\Lambda)$. Let $\mathcal{M}$ be the set of all measurable functions.

Let m be a functional on $\mathcal{M}$ with the following conditions.

(1) $0\leqq m(f)\leqq+\infty$ *for* $f\in\mathcal{M}$.

(2) $|f|\leqq|g|$ *a.e.* $\Rightarrow m(f)\leqq m(g)$.

(3) $m(f)=0$ *if and only if* $f=0$ *a.e.*

(4) $\inf(f,g)=0$ *i.e.* $f\cap g=0\Rightarrow m(f+g)=m(f)+m(g)$.

(5) $0\leqq f_n\uparrow$, $\sup_n m(f_n)<+\infty\Rightarrow m(f)=\sup_n m(f_n)$ *for* $f=\sup_n f_n$.

(6) $m(\alpha_nf)\to 0$ *as* $\alpha_n\to 0$ *for every* f *with* $m(f)<+\infty$.

(7) $m(\alpha f)\geqq\alpha m(f)$ *for* $1\geqq\alpha\geqq 0$.

(8) $m(\chi_E)<+\infty$ *for every characteristic function* χ_E *of* E *with* $\mu(E)<+\infty$.

We shall consider a subset of $\mathcal{M}$: $L_m=\{f\in\mathcal{M},\ m(f)<+\infty\}$. We shall identify f and g if $f=g$ a.e. in L_m. If $m(f)=\int|f|^pd\mu(0<p\leqq 1)$, then L_m coincides with L^p. If $\Omega=\bigcup_{i=1}^{\infty}A_i$ (disjoint union) $(0<\mu(A_i)<+\infty$ for all $i=1,2,\cdots)$ and $m(f)=\sum_{i=1}^{\infty}\frac{1}{2^i\mu(A_i)}\int_{A_i}\frac{|f|}{1+|f|}d\mu$, then L_m is the space of all measurable functions (essentially finite). In this case, (5) must be changed.

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Abstract form of L_m is considered by Nakano [2].

3. **Lemma 1.** *There exists a function $M(t, \omega)$ of two variables of real $t \geqq 0$ and $\omega \in \Omega$ such that*

1 *$M(t, \omega)$ is a non-negative measurable function of ω for a fixed $t \geqq 0$ with $M(0, \omega) = 0$,*

2 *$M(t, \omega)$ is a continuous function of $t \geqq 0$ for a.e. $\omega \in \Omega$,*

3 *$M(\alpha, \omega) \geqq M(\beta, \omega)$ for $\alpha \geqq \beta \geqq 0$ a.e. $\omega \in \Omega$,*

4 *$M(\alpha + \beta, \omega) \leqq M(\alpha, \omega) + M(\beta, \omega)$ for a.e. $\omega \in \Omega (\alpha \geqq 0, \beta \geqq o)$,*

5 *$m(f) = \int M(|f(\omega)|, \omega) d\mu$.*

Since m is an additive functional of L_m, Lemma 1 is essentially proved in [1] by virtue of Radon-Nikodym's theorem and Conditions (1) ~ (7).

Lemma 2. *$m(f+g) \leqq m(f) + m(g)$ for $f, g \in L_m$.*

This lemma is a direct consequence of Lemma 1.

Lemma 3. *L_m is a topological vector space with the topology by m (sequential topology).*

This lemma follows from Lemma 2 and (6). Moreover the topology by m is complete by (5) and (6).

Remark. L_m is not locally convex in general. It may happen that the dual of L_m consists of only zero element.

Lemma 4. *$\lim_{n \to \infty} m(f_n - f) = 0$ implies that f_n converges to f in measure in every measurable set of finite measure.*

Proof. For a measurable set E with finite measure and $\delta > 0$, there exist a positive number α and a measurable set A such that $\mu(A) < \delta$ and $\alpha \leqq M(1, \omega) < +\infty$ for $\omega \in E - A$ by Lemma 1 and (8). Hence, for $1 > \varepsilon > 0$ and $g \in L_m$

$$\mu(\omega \in E, |g(\omega)| > \varepsilon) \leqq \alpha \int_{|g| > \varepsilon} M\left(\frac{|g(\omega)|}{\varepsilon}, \omega\right) d\mu + \delta$$

$$\leqq \alpha m\left(\frac{g}{\varepsilon}\right) + \delta \leqq \frac{\alpha}{\varepsilon} m(g) + \delta.$$

Hence, $m(f_n) \to 0$ implies $\mu(\omega \in E, |f_n(\omega)| > \varepsilon) \to 0$ for $\varepsilon > 0$. This proves Lemma 4.

Lemma 5. *If $\{f_n\}$ is a C-sequence of L_m, then f_n converges to 0 a.e. on every measurable set of finite measure.*

This lemma is proved by Lemma 4 and the result of Kolmogorov-Khintchin (cf. [3]).

Theorem. *If $\{f_n\}$ is a C-sequence of L_m, then $\sum_{n=1}^{\infty} f_n$ is convergent.*

Proof. Suppose that $m(f_n) \geqq \delta > 0$ for all $f_n \in L_m (n = 1, 2, \cdots)$ and f_n converges to 0 a.e. on every measurable set of finite measure. For f_1, we choose some measurable set F with $\mu(F) < +\infty$ such that

$m(f_1\chi_F)\geqq\frac{\delta}{2}$. We can find $\varepsilon>0$ such that $m(f_1\chi_{E_0})\leqq\frac{\delta}{4}$ for $\mu(E_0)<\varepsilon$, $E_0\subset F$. Since f_n converges to 0 a.e. on F, there exists E with $\mu(E)<\varepsilon$ such that f_n converges uniformly on $F-E$ (by Egoroff's theorem) and $m(f_1\chi_E)\leqq\frac{\delta}{4}$. For $E_1=F-E$, we have $m(f_1\chi_{E_1})\geqq\frac{\delta}{4}$.

Since $\{f_n\}$ converges uniformly on E_1, we can find f_{n_2} such that $m(f_{n_2}\chi_{E_1})\leqq\frac{\delta}{4}$ by (6) and (8). By the same argument as above, there exists E_2 with $\mu(E_2)<+\infty$ and $E_1\cap E_2=\phi$ such that $m(f_{n_2}\chi_{E_2})\geqq\frac{\delta}{4}$ and $\{f_n\}$ converges uniformly on E_2.

Hence, by induction we can choose a sequence of numbers $1=n_1<n_2<\cdots$ and mutually disjoint measurable sets with finite measure $E_1, E_2, \cdots$ such that

$$m(f_{n_k}\chi_{E_k})\geqq\frac{\delta}{4}\ (k=1,2,\cdots).$$

By (4) and (7), we have

$$m\left(\sum_{k=1}^{l}\frac{1}{k}f_{n_k}\right)\geqq m\left(\sum_{k=1}^{l}\frac{1}{k}f_{n_k}\chi_{E_k}\right)\geqq\sum_{k=1}^{l}\frac{1}{k}m(f_{n_k}\chi_{E_k})$$
$$\geqq\frac{\delta}{4}\sum_{k=1}^{l}\frac{1}{k}\ \text{for all } l=1,2,\cdots.$$

Hence, $\{f_{n_k}\}$ is not a C-sequence. This prove that $\lim_{n\to\infty}m(f_n)=0$ for a C-sequence $\{f_n\}$ by Lemma 5. It is easy to see that if $\lim_{n\to\infty}m(f_n)=0$, then $\sum_{n=1}^{\infty}f_n$ is convergent for a C-sequence $\{f_n\}$ (cf. [3]).

References

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