# 119. *On the Bi-ideals in Associative Rings*

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By a ring we shall mean an arbitrary associative ring. For the terminology not defined here we refer to N. Jacobson [3] and N. H. McCoy [8]. We announce some properties of bi-ideals in rings which are analogous to some properties of bi-ideals in semigroups.

For the subsets $X$ and $Y$ of a ring $A$ by the product $XY$ we mean the subring of $A$ which is generated by the set of all products $xy$, where $x \in X$, $y \in Y$. By a bi-ideal $B$ of $A$ we mean a subring $B$ of $A$ satisfying the condition

$$BAB \subseteq B. \tag{1}$$

Obviously every one-sided ideal of $A$ is a bi-ideal and the intersection of a left and a right ideal of $A$ is also a bi-ideal. It may be remarked that the notion of the bi-ideal in semigroups is a special case of the $(m, n)$-ideal introduced by S. Lajos [4]. The notion of bi-ideal for associative rings was earlier mentioned by S. Lajos [5]. He noted that the set of all bi-ideals of a regular ring is a multiplicative semigroup. The concept of the bi-ideal was introduced by R. A. Good and D. R. Hughes [1]. An interesting particular case of the bi-ideal is the notion of quasi-ideal due to O. Steinfeld [9] which is defined as follows. A submodule $Q$ of a ring $A$ is called a quasi-ideal of $A$ if the following condition holds:

$$AQ \cap QA \subseteq Q. \tag{2}$$

It is known that the product of any two quasi-ideals is a bi-ideal (see S. Lajos [5]). It may be remarked that in case of regular rings the notions of bi-ideal and quasi-ideal coincide.

In the following we formulate some general properties of bi-ideals in rings and characterize two important classes of rings in terms of bi-ideals.

**Proposition 1.** *The intersection of an arbitrary set of bi-ideals $B_i$ $(i \in I)$ of a ring $A$ is again a bi-ideal of $A$.*

**Proposition 2.** *The intersection of a bi-ideal $B$ of a ring $A$ and a subring $S$ of $A$ is a bi-ideal of the ring $S$.*

**Proposition 3.** *For an arbitrary subset $T$ of a ring $A$ and for a bi-ideal $B$ of $A$ the products $BT$ and $TB$ are bi-ideals of $A$.*

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In analogy with the case of semigroups (cf. S. Lajos [7]) we obtain the following result.

**Proposition 4.** *Let $B$ be an arbitrary bi-ideal of the ring $A$, and $C$ be a bi-ideal of the ring $B$ such that $C^2=C$. Then $C$ is a bi-ideal of $A$.*

**Proposition 5.** *An arbitrary associative ring $A$ contains no non-trivial bi-ideal if and only if $A$ is either a zero ring of prime order or else a division ring.*

**Proposition 6.** *Let $T$ be a non-empty subset of the ring $A$. Then the bi-ideal $T_{(1,1)}$ of $A$ generated by $T$ is of the form*

$$T_{(1,1)} = IT + IT^2 + TAT, \tag{3}$$

*where $I$ denotes the ring of rational integers.*

**Proposition 7.** *For any associative ring $A$ denote $\bar{A}$ the set of all subrings of $A$, and $A_1$ the set of all bi-ideals of $A$. Then $\bar{A}$ and $A_1$ are semigroups under the multiplication of subsets (defined in the introduction) and $A_1$ is a two-sided ideal of $\bar{A}$.*

**Remark.** The multiplicative semigroup of all non-empty subsets of an arbitrary semigroup was formerly investigated by S. Lajos [7]. He proved, among others, that the set of all the bi-ideals of a semigroup $S$ is a two-sided ideal of the multiplicative semigroup of all non-empty subsets of $S$.

The following result is in complete analogy with a semigroup-theoretical theorem of S. Lajos [4].

**Theorem 1.** *For an arbitrary non-empty subset $B$ of an associative ring $A$ the following conditions are pairwise equivalent:*

(i) *$B$ is a bi-ideal of $A$.*

(ii) *$B$ is a left ideal of a right ideal of $A$.*

(iii) *$B$ is a right ideal of a left ideal of $A$.*

**Theorem 2.** *For an associative ring $A$ the following conditions are equivalent:*

( I ) *$A$ is regular.*

( II ) *$L \cap R = RL$ for every left ideal $L$ and every right ideal $R$ of $A$.*

( III ) *For any elements $a$, $b$ of $A$.*

$$(a)_r \cap (b)_l = (a)_r (b)_l.$$

( IV ) *For any element $a$ of $A$*

$$(a)_r \cap (a)_l = (a)_r (a)_l.$$

( V ) *$(a)_{(1,1)} = (a)_r (a)_l$ for any element $a$ of $A$.*

( VI ) *$(a)_{(1,1)} = aAa$ for any element $a$ of $A$.*

(VII) *$QAQ = Q$ for every quasi-ideal $Q$ of $A$.*

(VIII) *$BAB = B$ for each bi-ideal $B$ of $A$.*

The equivalence of the above conditions (I)–(VI) in case of semigroups was proved by S. Lajos [6] and K. Iséki [2].

**Theorem 3.** *The following conditions for an associative ring A are equivalent*:

( I ) *A is a regular two-sided ring.*[1]

( II ) *A is a subcommutative regular ring.*[2]

( III ) *A is strongly regular.*

( IV ) $B^2=B$ *for any bi-ideal B of A.*

( V ) $Q^2=Q$ *for any quasi-ideal Q of A.*

( VI ) $L\cap R=LR$ *for every left ideal L and every right ideal R of A.*

(VII) $L\cap L'=LL'$ *for any two left ideals* $L, L'$ *of A.*

(VIII) *A is regular and it is a subdirect sum of division rings.*

( IX ) *A is a regular ring with no non-zero nilpotent elements.*

( X ) *The multiplicative semigroup of A is a semilattice of groups.*

## References

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1) An associative ring $A$ is said to be two-sided (or duo) if every one-sided (left or right) ideal of $A$ is two-sided.

2) An associative ring $A$ is called subcommutative if $aA=Aa$ for any element $a$ of $A$.