111. *Rings in which Every Maximal Ideal is generated by a Central Idempotent**)

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Introduction. Recently, in his paper [2], M. Satyanarayana has proved that, for a commutative ring R with identity 1, the following conditions are equivalent:

(1) *R is a finite direct sum of fields.*
(2) *Every maximal ideal is generated by an idempotent.*
(3) *Every maximal ideal is a direct summand of R.*
(4) *Every maximal ideal is R-projective as a right R-module and is principally generated by a zero-divisor.*
(5) *Every proper maximal ideal is R-injective as a right R-module.*
(6) *R has no nilpotents and every proper maximal ideal has a non-zero annihilator.*

By using the technique of the sheaf theory as in [1], we shall extend the above result to a non-commutative case.

In this paper all rings R are assumed to possess an identity element 1, and all R-modules are unitary modules. The term "ideals" will always mean "two-sided ideals".

1. Preliminaries. Pierce [1] defined, for each ring R, a sheaf $S(R)$ of rings over a Boolean space $X(R)$ (that is, a totally disconnected compact Hausdorff space) in such a way that R is the ring of global cross sections of $S(R)$.

Let $B(R)$ be the Boolean ring consisting of all central idempotents of R and let $X(R)$ be the *Spec* $B(R)$ consisting of all prime ideals of $B(R)$. Let x be a point in $X(R)$. Then, for each element e in x, there is a neighborhood of x, namely $U_e(x)=\{y \in X(R) \mid e \in y\}$. These neighborhoods form a base of the open sets of $X(R)$ and with this topology $X(R)$ becomes a Boolean space. Note that the neighborhood $U_e(x)$ is an open-closed set of $X(R)$.

For x in $X(R)$, we denote R/Rx, by R_x, where Rx is the ideal of R generated by x. Define $S(R)=\bigcup_{x\in X(R)} R_x$. Let $\pi: S(R)\to X(R)$ be given by the condition $\pi^{-1}(x)=R_x$. For $r\in R$ and $x\in X(R)$, let $\sigma_r(x)$ be the image of r under the natural homomorphism of R onto R_x.

*) Dedicated to Professor K. Asano for the celebration of his sixtieth birthday.

Then $S(R)$ is a topological space with the open sets $\sigma_r(U_e(x))$ for all $e \in B(R)$, all $x \in X(R)$ and all $r \in R$, and also it becomes a sheaf of rings over the Boolean space $X(R)$.

If we denote the ring of all sections of $S(R)$ over $X(R)$ by $\Gamma(X(R), S(R))$, then the mapping $r \rightarrow \sigma_r$ is a ring isomorphism of R onto $\Gamma(X(R), S(R))$ (Theorem 4.4 of [1]).

Let U be an open set in $X(R)$. If we will use the following notation

$$S(R)_U = \bigcup_{x \in U} R_x \cup \{0_x \mid x \in X(R)\}$$

where 0_x is the zero element in R_x, then $S(R)_U$ is a sheaf of rings over $X(R)$. Especially, if N is an open-closed set, then $S(R)$ is isomorphic to $S(R)_N \oplus S(R)_{X(R)-N}$, namely, for each x in $X(R)$, there is a ring isomorphism f_x of R_x onto the direct sum of the stalk of $S(R)_N$ over x and the stalk of $S(R)_{X(R)-N}$ over x, such that the extension mapping

$$\bigcup_{x \in X(R)} f_x : S(R) \rightarrow S(R)_N \oplus S(R)_{X(R)-N}$$

is a homeomorphism. Therefore $\Gamma(X(R), S(R))$ is isomorphic to the direct sum of $\Gamma(X(R), S(R)_N)$ and $\Gamma(X(R), S(R)_{X(R)-N})$ as rings (Proposition 7.10 of [1]).

Let R be a biregular ring, that is, each principal ideal in R is generated by a central idempotent. Then, the induced sheaf $S(R)$ is a sheaf of simple rings over the Boolean space $X(R)$. Conversely, the ring of global cross sections of a sheaf of simple rings over a Boolean space is a biregular ring ([1], § 11).

In [3], Villamayor and Zelinsky also discussed the sheaf $S(R)$ induced by a commutative ring R. However, their following results hold for a non-commutative case.

Let M be a right R-module. For a point x in $X(R)$, we define $M_x = M \otimes_R R_x = M/Mx$, where Mx is the submodule of M generated by x, and for a in M, we denote by a_x the image of a under the natural homomorphism: $M \rightarrow M_x$. Similarly, we can define ${}_xM$ for a left R-module M. Then,

[a] *R_x is flat as an R-module.*

We shall give an elementary proof of the above well known result. Indeed, for right R-modules M and N, consider an arbitrary monomorphism f from N into M. Then, we may show that the induced homomorphism f_x from N_x into M_x is also a monomorphism. If $f_x(n_x) = 0_x$ for an element n_x in N_x, then $f(n) \in Mx$. Hence $f(n) = me$ for some m in M and e in x, and hence $f(n - ne) = 0$. Since f is a monomorphism, $n - ne = 0$. Thus we have $n_x = (ne)_x = 0$.

[b] *For any right R-module M, every finite set of elements in Mx is contained in Me for some e in x. If $m_x = 0_x$ for all m in a finite*

subset of M, then $m(1-e)=0$ *for some e in x and for all m in the subset.*

[c] *Let a and b be elements of any right R-module M and* $a_x=b_x$ *at a point x in* $X(R)$. *Then* $a(1-e)=b(1-e)$ *for some e in x. If* $a_x=b_x$ *for all x in* $X(R)$, *then* $a=b$.

[d] *Let M be a right R-module. If N is a submodule of M and* $N_x=M_x$ *for all x in* $X(R)$ (*note that, since* R_x *is flat over R, we have* $N_x\subseteq M_x$), *then* $N=M$. *If M is finitely generated and* $N_x=M_x$ *for a point x in* $X(R)$, *then there is an element e in x such that* $N(1-e)=M(1-e)$.

[e] *If* $X(R)$ *is covered by a finite number of the* $U_e(x)$, *say* $U_{e_1}, U_{e_2}, \cdots, U_{e_n}$, *then there exist* $c_{e_1}, c_{e_2}, \cdots, c_{e_n}$ *in the center of R such that*

$$1=\sum_{i=1}^{n}(1-e_i)c_i.$$

Indeed, it is easily seen from $X(R)=\bigcup_{i=1}^{n} U_{e_i}$ that the ideal of $B(R)$ generated by $1-e_i$ $(i=1,2,\cdots,n)$ is $B(R)$. Hence $1\in\sum_{i=1}^{n}(1-e_i)B(R)$, so that 1 is a linear combination in $B(R)$ of $1-e_i (i=1,2,\cdots,n)$. Such a linear combination is also a linear combination in the center of R of $1-e_i$ $(i=1,2,\cdots,n)$.

2. Biregular rings. We recall that a ring R is a biregular ring if every principal ideal of R is generated by a central idempotent. As is easily seen from Theorem 11.1 of [1], R is biregular if and only if Rx is a maximal ideal of R for all x in $X(R)$. Moreover, we have the following:

Proposition 2.1. *R is biregular if and only if every maximal ideal is generated by a set of central idempotents of R.*

Proof. Let x be a point in $X(R)$. Then Rx is a maximal ideal in R. To see this, let m be a maximal ideal in R containing Rx. Then the set $B(m)$ of all central idempotents in m is an ideal of $B(R)$ and $x=B(m)$. Thus $m=RB(m)=Rx$. This shows that Rx is a maximal ideal of R.

Conversely, it is evident that every maximal ideal of a biregular ring is generated by its central idempotents.

Proposition 2.2. *Let R be a ring and let x be a point in* $X(R)$. *Then, x is an isolated point in* $X(R)$ *if and only if Rx is generated by a central idempotent in x.*

Proof. If x is an isolated point, then $\{x\}$ is an open-closed set. It follows from Proposition 7.10, Theorem 8.4 and Lemma 15.1 of [1] that $\Gamma(X(R), S(R))$ is isomorphic to the direct sum of $\Gamma(X(R), S(R)_{\{x\}})$ and $\Gamma(X(R), S(R)_{X(R)-\{x\}})$ as rings. Let σ_r be as in §1. Then σ_r is contained in $\Gamma(X(R), S(R)_{X(R)-\{x\}})$ if and only if $\sigma_r(x)=0$, that is, $r\in Rx$.

Thus Rx is isomorphic to $\Gamma(X(R), S(R)_{X(R)-\{x\}})$, by Theorem 4.4 of [1], and hence Rx is generated by a central idempotent as desired.

Conversely, let $Rx=Re$ for some central idempotent e in x, and let U_e be the neighborhood of x for e. For any point y in U_e, since $Ry \supseteq Rx$, we have $y=x$. Thus $U_e=\{x\}$, and hence x is an isolated point.

Now, note that we can see that $R_x \otimes_R R$ is a right R_x-module, if we define $(s_x \otimes t) r_x = s_x \otimes tr$ for s, t and r in R. Then, we have

Proposition 2.3. *Let R be a ring and let M be a finitely generated right R-module. If x is an isolated point of $X(R)$, then the canonical mapping*

$$R_x \otimes_R \mathrm{Hom}_R(M, R) \to \mathrm{Hom}_R(M, R_x \otimes_R R) = \mathrm{Hom}_{R_x}(M_x, R_x)$$

is an isomorphism.

Proof. Since Rx is generated by a central idempotent, Rx is a projective right R-module. Hence, as is well known, the canonical mapping

$$R_x \otimes_R \mathrm{Hom}_R(M, R) \to \mathrm{Hom}_R(M, R_x \otimes_R R) = \mathrm{Hom}_{R_x}(M_x, R_x)$$

is an isomorphism.

Proposition 2.4. *Let R be a biregular ring such that $X(R)$ is discrete and R_x satisfies the minimum condition for right ideals for all x in $X(R)$, and let M be a finitely generated right R-module. Then M is a projective right R-module.*

Proof. Let x be a point in $X(R)$. Then the canonical mapping

$$R_x \otimes_R \mathrm{Hom}_R(M, R) \to \mathrm{Hom}_{R_x}(M_x, R_x)$$

is an isomorphism, by Proposition 2.3. Since R is a biregular ring, R_x is a simple ring. Hence M_x is a finitely generated projective right R_x-module, and there exist a finite set $\{f_i\}_{i \in I}$ in $\mathrm{Hom}_R(M, R)$ and a finite set $\{a_i\}_{i \in I}$ in M such that

$$m_x = \sum_{i \in I} a_{ix} f_{ix}(m_x) = (\sum_{i \in I} a_i f_i(m))_x$$

for all m in M. By [b] in §1, there is some e in X such that

$$m(1-e) = \sum_{i \in I} a_i f_i(m)(1-e)$$

for all m in M. Tnus by the partition property of $X(R)$, there exist a finite set $\{x_j\}$ in $X(R)$ and a finite set of neighborhoods $\{U_{e_j}(x_j)\}$, $j=1, 2, \cdots, n$, such that $X(R)$ is covered by $U_{e_1}(x_1)$, $U_{e_2}(x_2)$, $\cdots$, $U_{e_n}(x_n)$. For each j, as is mentioned above, there exist a finite set $\{f_i^j\}_{i \in I(j)}$ in $\mathrm{Hom}_R(M, R)$ and a finite set $\{a_i^j\}_{i \in I(j)}$ in M such that

$$m(1-e_j) = \sum_{i \in I(j)} a_i^j f_i^j(m)(1-e_j)$$

for all m in M. Since $X(R) = \bigcup^n U_{e_j}(x_j)$, it follows from [d] in §1 that there exist $c_1, c_2, \cdots, c_n$ in the center of R such that $1 = \sum_{j=1}^n (1-e_j) c_j$. Then, for every m in M,

$$\begin{aligned} m=m1=m\sum_{j=1}^{n}(1-e_j)c_j \\ =\sum_{j=1}^{n}m(1-e_j)c_j \\ =\sum_{j=1}^{n}\sum_{i\in I(j)}a_i^j f_i^j(m)(1-e_j)c_j \\ =\sum_{j=1}^{n}\sum_{i\in I(j)}(a_i^j(1-e_j)c_j)f_i^j(m). \end{aligned}$$

Thus M is a finitely generated projective right R-module.

From Propositions 2.1, 2.2 and 2.4, we obtain the following main theorem:

Theorem 2.5. *Let R be a ring such that, for each $x \in X(R)$, R_x satisfies the minimum condition for right ideals, then the following conditions are equivalent:*

(1) *R is a semi-simple Artinian ring.*

(2) *Every ideal of R is generated by a central idempotent.*

(3) *Every maximal ideal of R is generated by a central idempotent.*

(4) *R is a biregular ring such that $X(R)$ is discrete.*

Proof. It is evident that (1) implies (2) and (2) implies (3). By Propositions 2.1 and 2.2, we have that (3) implies (4). And by Proposition 2.4, we have that (4) implies (1).

3. SRD-rings. Now, an ideal I of a ring R is called a strongly right direct summand of R if I is a direct summand of R as a ring whenever I is a direct summand of R as a right R-module. And we shall call R an *SRD-ring* if every ideal of R is a strongly right direct summand of R. Then we have

Proposition 3.1. *Let R be an SRD-ring, and let m be a maximal ideal of R.*

(1) *If m is projective as a right R-module and is generated by a central zero divisor, then m is generated by a central idempotent.*

(2) *If m is injective as a right R-module, then m is generated by a central idempotent.*

Proof. (1) Let $m=cR$, c being a central zero-divisor. Then c^*, the left annihilator of c, is a non-zero ideal. Consider the exact sequence

$$0 \longrightarrow c^* \xrightarrow{\ i\ } R \xrightarrow{\ j\ } cR \longrightarrow 0,$$

where i is the inclusion mapping and $j: r \to cr$ for $r \in R$. Since cR is projective, the above sequence splits. Hence $c^*=eR$ for some central idempotent e, since R is an *SRD*-ring. Now $0=ce$. Therefore $c=c(1-e)$ and $m \subseteq (1-e)R$. Since m is a maximal ideal and $e \neq 0$, we obtain $m=(1-e)R$. Thus m is generated by a central idempotent.

(2) This is evident.

Proposition 3.2. *Let R be a ring without non-zero nilpotent elements, and let m be a maximal ideal of R such that its right annihilator m^* is not zero. Then m is generated by a central idempotent.*

Proof. Since R has no non-zero nilpotent elements and m is maximal, $R=m\oplus m^*$ and hence m is generated by a central idempotent.

Note that every commutative ring is an *SRD*-ring, and that if R is a commutative biregular ring, then R_x satisfies the minimum condition for all x in $X(R)$. Thus, from Theorem 2.5, Propositions 3.1 and 3.2, we have

Corollary 3.3 (M. Satyanarayana). *Let R be a commutative ring. Then the conditions mentioned in Introduction are equivalent to one another.*

References

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