153. Absolute Summability by Logarithmic Method of Fourier Series

By Masako IZUMI and Shin-ichi IZUMI

(Comm. by Kinjirô KUNUGI, M. J. A., Sept. 12, 1970)

1. Introduction and Theorems.

1.1. Let $\sum a_n$ be an infinite series and (s_n) be the sequence of partial sums. If the function

$$L(x)=\frac{-1}{\log(1-x)}\sum_{n=1}^{\infty}\frac{s_n x^n}{n} \tag{1}$$

is of bounded variation on an interval $(c,1)$, then the series $\sum a_n$ is said to be absolutely summable by logarithmic method or $|L|$-summable (see [1] and [2]).

Let f be an even integrable function with period 2π and its Fourier series be $\sum a_n \cos nx$. R. Mohanty and J. N. Patnaik [2] have proved the following

Theorem 1. *If the function*

$$\frac{1}{t\log(2\pi/t)}\int_t^{\pi}\frac{f(u)du}{2\sin u/2}=\frac{g(t)}{t\log(2\pi/t)} \tag{2}$$

is integrable in the interval $(0,\pi)$, *then the Fourier series of* f *is* $|L|$*-summable at the origin.*

Our first object of this paper is to give an alternative proof of this theorem.

1.2. Let (p_n) be a sequence of non-negative numbers such that

$$p(x)=\sum_{n=1}^{\infty}p_n x^n<\infty \quad \text{for} \quad 0<x<1.$$

If the function

$$P(x)=\frac{1}{p(x)}\sum_{n=1}^{\infty}p_n s_n x^n \tag{3}$$

is of bounded variation on an interval $(c,1)$ $(0<c<1)$, then we say that the series $\sum a_n$ is absolutely Perron summable or $|P|$-summable. According as $p_1=1$ or $p_n=1/n$, then $|P|$-summability reduces to $|A|$-summability or $|L|$-summability, respectively.

Theorem 1 is generalized as follows:

Theorem 2. *Suppose that* (i) *the sequence* (np_n) *is of bounded variation and that* (ii) *there is an* a, $0<a<1$, *such that*

$$(1-x)^a p(x)\downarrow \quad as \quad x\uparrow 1. \tag{4}$$

If $g(t)/tp(1-t)$ *is integrable in the interval* $(0,\pi)$, *then the Fourier series of* f *is* $|P|$*-summable at the origin.*

From the proof of Theorem 2, we can see that the condition (i) may be replaced by that

$$p'(z)=O(1/|1-z|), \quad p''(z)=O(1/|1-z|^2) \quad \text{as} \quad z\to 1$$

where $z=xe^{it}$ and $p(z)=\sum p_n z^n$.

If $p_n=1/n$, then Theorem 2 reduces to Theorem 1.

2. Proof of Theorems.

2.1. Proof of Theorem 1.

Let s_n be the n th partial sum of Fourier series of f at the origin, then

$$(6) \qquad \frac{\pi}{2}s_n=\int_0^\pi f(t)\frac{\sin(n+1/2)t}{2\sin t/2}dt=(n+1/2)\int_0^\pi g(t)\cos(n+1/2)t\,dt$$

where $g(t)$ is defined by (2). By the definition (1),

$$\begin{aligned}\frac{\pi}{2}L(x)&=\frac{-1}{\log(1-x)}\sum_{n=1}^\infty\left(1+\frac{1}{2n}\right)x^n\int_0^\pi g(t)\cos(n+1/2)t\,dt\\&=\frac{-1}{\log(1-x)}\int_0^\pi g(t)\left(\sum_{n=1}^\infty x^n\cos(n+1/2)t\right)dt\\&\quad+\frac{-1}{2\log(1-x)}\int_0^\pi g(t)\left(\sum_{n=1}^\infty\frac{x^n\cos(n+1/2)t}{n}\right)dt\\&=M(x)+N(x).\end{aligned}$$

We shall first prove that $M(x)$ is of bounded variation on the interval $(c,1)$. Since

$$\begin{aligned}\sum_{n=1}^\infty x^n\cos(n+1/2)t&=\Re\left(\sum_{n=1}^\infty x^n e^{i(n+1/2)t}\right)\\&=\Re\left(e^{it/2}\sum_{n=1}^\infty x^n e^{int}\right)=\Re(xe^{3it/2}/(1-xe^{it}))\end{aligned}$$

and

$$|1-xe^{it}|^2=(1-x\cos t)^2+x^2\sin^2 t=(1-x)^2+4\sin^2 t/2,$$

we have

$$\begin{aligned}\int_c^1|M'(x)|dx&\leqq\int_0^\pi|g(t)|\,dt\int_c^1\left|\frac{d}{dx}\left(\frac{x}{(1-xe^{it})\log(1-x)}\right)\right|dx\\&=\int_0^\pi|g(t)|\,dt\int_c^1\left|\frac{1}{(1-xe^{it})^2\log(1-x)}+\frac{x}{(1-x)(1-xe^{it})(\log(1-x))^2}\right|dx\\&\leqq A\int_0^\pi|g(t)|\,dt\left(-\int_c^{1-t}\frac{dx}{(1-x)^2\log(1-x)}-\frac{1}{t^2}\int_{1-t}^1\frac{dx}{\log(1-x)}\right.\\&\qquad\left.+\frac{1}{t}\int_{1-t}^1\frac{dx}{(1-x)(\log(1-x))^2}\right)\leqq A\int_0^\pi\frac{|g(t)|}{t\log(2\pi/t)}dt\leqq A.\end{aligned}$$

Concerning $N(x)$,

$$\begin{aligned}N'(x)&=\frac{-1}{2\log(1-x)}\int_0^\pi g(t)\left(\sum_{n=1}^\infty x^{n-1}\cos(n+1/2)t\right)dt\\&\quad+\frac{-1}{(1-x)(\log(1-x))^2}\int_0^\pi g(t)dt\int_0^x\left(\sum_{n=1}^\infty u^{n-1}\cos(n+1/2)t\right)du\end{aligned}$$

and then the total variation of $N(x)$ is

$$\int_c^1 |N'(x)|\,dx \leqq \int_c^1 \frac{dx}{|\log(1-x)|}\int_0^\pi \frac{|g(t)|}{|1-xe^{it}|}dt$$
$$+\int_c^1 \frac{dx}{(1-x)(\log(1-x))^2}\int_0^\pi |g(t)|\,dt\int_0^x \frac{du}{|1-ue^{it}|}$$
$$\leqq A\int_0^\pi |g(t)|\,dt\left(\int_c^1 \frac{dx}{|1-xe^{it}|\,|\log(1-x)|}+A\right)$$

Thus the theorem is proved.

2.2. Proof of Theorem 2.

By (3) and (6),

$$\frac{\pi}{2}P(x)=\frac{1}{p(x)}\sum_{n=1}^\infty (n+1/2)p_n x^n \int_0^\pi g(t)\cos(n+1/2)t\,dt$$
$$=\frac{1}{p(x)}\int_0^\pi g(t)\left(\sum_{n=1}^\infty (n+1/2)p_n x^n \cos(n+1/2)t\right)dt.$$

We put $p(z)=\sum p_n z^n$, for complex z, then

$$\sum_{n=1}^\infty (n+1/2)p_n x^n \cos(n+1/2)t=\mathfrak{R}\left(xe^{it/2}p'(xe^{it})+\frac{1}{2}e^{it/2}p(xe^{it})\right)$$

where $'$ denotes the differentiation with respect to x. Hence

$$\int_c^1 |P'(x)|\,dx \leqq \int_0^\pi |g(t)|\,dt\int_c^1 \left|\frac{d}{dx}\left(\frac{xp'(xe^{it})+p(xe^{it})/2}{p(x)}\right)\right|dx.$$

It is enough to prove that

$$(7)\qquad \int_c^1 \left|\frac{d}{dx}\left(\frac{xp'(xe^{it})+p(xe^{it})/2}{p(x)}\right)\right|dx=\int_c^1 |q(x)|\,dx \leqq \frac{A}{tp(1-t)},$$

where

$$q(x)=\frac{(1+e^{it}/2)p'(xe^{it})+xe^{it}p''(xe^{it})}{p(x)}-\frac{(xp'(xe^{it})+p(xe^{it})/2)/p'(x)}{(p(x))^2}.$$

By the condition (4), we get

$$\int_c^{1-t}\frac{|p'(xe^{it})|}{p(x)}dx \leqq \int_c^{1-t}\frac{dx}{(1-x)p(x)} \leqq \frac{A}{t^a p(1-t)}\int_c^{1-t}\frac{dx}{(1-x)^{1-a}} \leqq \frac{A}{p(1-t)},$$
$$\int_c^{1-t}\frac{|p''(xe^{it})|}{p(x)}dx \leqq A\int_c^{1-t}\frac{dx}{(1-x)^2 p(x)}$$
$$+A\int_c^{1-t}\frac{dx}{(1-x)p(x)\log 1/x} \leqq \frac{A}{tp(1-t)}$$

and

$$\int_c^{1-t}\frac{|xp'(xe^{it})+p(xe^{it})/2|\,p'(x)}{(p(x))^2}dx \leqq A\int_c^{1-t}\frac{p'(x)}{(1-x)(p(x))^2}dx$$
$$\leqq \frac{A}{t^a p(1-t)}\int_x^{1-t}\frac{dx}{(1-x)^{2-a}} \leqq \frac{A}{tp(1-t)}.$$

Combining above three inequalities, we get

$$(8)\qquad \int_c^{1-t}|q(x)|\,dx \leqq \frac{A}{tp(1-t)}.$$

On the other hand, we have

(9)
$$p'(xe^{it})=\sum_{n=1}^{\infty} np_n x^{n-1}e^{int}=e^{it}\sum_{n=1}^{\infty}\Delta(np_n)\frac{1-x^n e^{int}}{1-xe^{it}} + \lim_{n\to\infty}(np_n)\frac{e^{it}}{1-xe^{it}}$$

and

(10)
$$\begin{aligned} p''(xe^{it}) &= \sum_{n=2}^{\infty} n(n-1)p_n x^{n-2}e^{int} \\ &= \sum_{n=2}^{\infty}\Delta(np_n)\sum_{m=2}^{n}(m-1)x^{m-2}e^{imt}+\lim_{n\to\infty} np_n\sum_{m=2}^{\infty}(m-1)x^{m-2}e^{imt} \\ &= \sum_{n=2}^{\infty}\Delta(np_n)\frac{d}{dx}\left(\sum_{m=1}^{n} x^{m-1}e^{imt}\right)+\lim_{n\to\infty} np_n\frac{d}{dx}\left(\sum_{m=1}^{\infty} x^{m-1}e^{imt}\right) \\ &= \sum_{n=2}^{\infty}\Delta(np_n)\frac{d}{dx}\left(\frac{1-x^n e^{int}}{1-xe^{it}}\right)e^{it}+\lim_{n\to\infty} np_n\frac{d}{dx}\left(\frac{e^{it}}{1-xe^{it}}\right) \\ &= e^{it}\sum_{n=2}^{\infty}\Delta(np_n)\left(\frac{(1-x^n e^{int})e^{it}}{(1-xe^{it})^2}-\frac{nx^{n-1}e^{int}}{1-xe^{it}}\right) \\ &\quad + \lim_{n\to\infty} np_n\frac{e^{2it}}{(1-xe^{it})^2}. \end{aligned}$$

By positivity of p_n, (9) and (10), we get

$$\int_{1-t}^{1}\frac{|p'(xe^{it})|}{p(x)}dx \leqq \frac{1}{p(1-t)}\int_{1-t}^{1}|p'(xe^{it})|\,dx \leqq \frac{A}{p(1-t)}$$
$$\int_{1-t}^{1}\frac{|p''(xe^{it})|}{p(x)}dx \leqq \frac{1}{p(1-t)}\int_{1-t}^{1}|p''(xe^{it})|\,dx \leqq \frac{A}{tp(1-t)}$$

and

$$\int_{1-t}^{1}\frac{|xp'(xe^{it})+p(xe^{it})/2|}{(p(x))^2}p'(x)dx \leqq \frac{A}{t}\int_{1-t}^{1}\left|\left(\frac{1}{p(x)}\right)'\right|dx \leqq \frac{A}{tp(1-t)}.$$

Combining above three estimations, we get

(11)
$$\int_{1-t}^{1}|q(x)|\,dx \leqq \frac{A}{tp(1-t)}.$$

The inequalities (8) and (11) give the required inequality (7). Thus Theorem 2 is proved.

References

[1] D. Borwein: A logarithmic method of summability. J. London Math. Soc., **33**, p. 212–220 (1958).
[2] R. Mohanty and J. N. Patnaik: On the absolute L- summability of a Fourier series. ibid., **43**, p. 452–456 (1968).