150. *Absolute Nörlund Summability Factor of Fourier Series*

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1. Let $\sum_{n=0}^{\infty} a_n$ be an infinite series and let (s_n) be the sequence of its partial sums. Let $(p_n)=(p_0, p_1, \cdots)$ be a sequence of positive numbers and $P_n=p_0+p_1+\cdots+p_n$ $(n=0,1,2,\cdots)$, $p_{-1}=P_{-1}=0$. We write

$$t_n=P_n^{-1}\sum_{k=0}^{n} p_{n-k}s_k=P_n^{-1}\sum_{k=0}^{n} P_{n-k}a_k \qquad (n=1,2,\cdots)$$

which is called the nth Nörlund mean of the series $\sum a_n$ or the sequence (s_n). If the sequence (t_n) is of bounded variation, then the series $\sum a_n$ is called to be absolutely summable (N, p_n) or summable $|N, p_n|$ and we write $\sum a_n \in |N, p_n|$.

Let f be an integrable function over the interval $(0, 2\pi)$ and be periodic with period 2π. We denote its Fourier series by

$$f(t)\sim \frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n \cos nt+b_n \sin nt)=\sum_{n=0}^{\infty} A_n(t).$$

The sequence (m_n) is called the absolute Nörlund summability factor or the $|N, p_n|$ summability factor of the Fourier series of f at the point x if $\sum m_n A_n(x) \in |N, p_n|$.

We suppose always that all m_n are non-negative.

S. V. Kolhekar [1] has proved the

Theorem A. *Let (m_n) be a monotone decreasing sequence satisfying the condition*

$$(1) \qquad \sum_{n=1}^{\infty} m_n n^{-1} \log n<\infty$$

and let (p_n) be a monotone increasing sequence such that

$$(2) \qquad p_n/P_n=O(1/n), \qquad \Delta(P_n/p_n)=O(1) \qquad \text{as } n\to\infty.$$

Then, if

$$(3) \qquad \Phi(t)=\int_0^t |\varphi(u)|\,du=O(t) \qquad \text{as } t\to 0$$

where $\varphi(u)=\varphi_x(u)=f(x+u)+f(x-u)-2f(x)$, *then* $\sum m_n A_n(x) \in |N, p_n|$.

We define a function $m(t)$ continuous on the interval $(1, \infty)$ such that $m(n)=m_n$ for $n=1,2,\cdots$ and $m(t)$ is linear for every non-integral t. Similarly $p(t)$ is defined by the sequence (p_n) and we put $P(t)=\int_0^t p(u)du$.

L. Leindler [2] has proved the following

Theorem B. *Let* (m_n) *and* (p_n) *be monotone decreasing sequences. If*

$$\sum_{n=1}^{\infty} m_n P_n^{-1} < \infty \quad \text{and} \quad \Phi(t) = O\left(m\left(\frac{1}{t}\right) \Big/ P\left(\frac{1}{t}\right)\right) \quad \text{as } t \to 0$$

or

$$\sum_{n=1}^{\infty} m_n P_n^{-1} \log\log n < \infty \quad \text{and} \quad \Phi(t) = O\left(t/\log\frac{1}{t}\right) \quad \text{as } t \to 0,$$

then $\sum m_n A_n(x) \in |N, p_n|$.

2. We have the following generalizations.

Theorem 1. *Let* (m_n) *be a monotone decreasing sequence satisfying the condition* (1) *and let* (p_n) *be a monotone increasing sequence such that*

$$\sum_{n=j+1}^{\infty} \frac{p_{n-j} - p_{n-j-1}}{P_{n-1}} \leqq \frac{A}{j} \quad \text{for all } j \geqq 1. \tag{4}$$

If the condition (3) *is satisfied, then* $\sum m_n A_n(x) \in |N, p_n|$.

The condition (2) implies the condition (4) and then Theorem 1 is a generalization of Theorem A.

Theorem 2. *Let* (m_n) *and* (p_n) *be monotone decreasing sequences satisfying the condition*

$$\sum_{n=1}^{\infty} m_n P_n^{-1} \log n < \infty.$$

If the condition (3) *is satisfied, then* $\sum m_n A_n(x) \in |N, p_n|$.

3. We can generalize Theorem 1 in the following form.

Theorem 3. *Let* (m_n) *be a monotone decreasing sequence and* (p_n) *be a monotone increasing sequence satisfying the condition* (4). *If*

$$\int_0^{\pi} \frac{\Phi(t)}{t^2}\, m\left(\frac{1}{t}\right) \log\frac{2\pi}{t}\, dt < \infty$$

and

$$\int_0^{\pi} \frac{\Phi(t)}{t^2}\, dt \int_0^{t} \frac{m(1/u)}{u}\, du < \infty,$$

then $\sum m_n A_n(x) \in |N, p_n|$.

This theorem has the following corollaries.

Corollary 1. *Suppose that* (m_n) *is a monotone decreasing sequence and that* (p_n) *is a monotone increasing sequence satisfying the condition* (4). *If*

$$\Phi(t) \leqq At \Big/ \left(\log\frac{1}{t}\right)^{\alpha} \quad \text{as } t \to 0$$

for an α, $0 \leqq \alpha \leqq 1$, *and*

$$\sum_{n=2}^{\infty} \frac{m_n (\log n)^{1-\alpha}}{n} < \infty \quad \text{or} \quad \sum_{n=3}^{\infty} \frac{m_n \log\log n}{n} < \infty$$

according as $0 \leqq \alpha < 1$ *or* $\alpha = 1$, *then* $\sum m_n A_n(x) \in |N, p_n|$.

Corollary 2. *Suppose that* (m_n) *is a monotone decreasing sequence and that* (p_n) *is a monotone increasing sequence satisfying the condition* (4). *If*

(5) $$\int_0^\pi \Phi(t)t^{-2}dt<\infty \quad \text{and} \quad \sum_{n=1}^\infty (m_n/n)<\infty,$$

then $\sum m_nA_n(x)\in|N,p_n|$.

The first condition of (5) is satisfied when $\Phi(t)\leqq At\Big/\left(\log\dfrac{1}{t}\right)^\alpha (\alpha>1)$ as $t\to 0$ or $\Phi(t)\leqq A\, t\, m(1/t)$ as $t\to 0$.

Theorem B is generalized as follows:

Theorem 4. *Suppose that* (m_n) *and* (p_n) *are monotone decreasing sequences. If*

$$\int_0^\pi \frac{\Phi(t)}{t^2}dt\int_0^t \frac{m(1/v)}{v^2P(1/v)}dv<\infty,$$

then $\sum m_nA_n(x)\in|N,p_n|$.

As a corollary of Theorem 4, we get

Corollary 3. *Suppose that* (m_n) *and* (p_n) *are monotone decreasing sequences. If*

$$\Phi(t)\leqq At\Big/\left(\log\frac{1}{t}\right)^\alpha \quad \textit{as} \quad t\to 0 \quad \textit{and} \quad \sum_{n=1}^\infty m_nP_n^{-1}(\log n)^{1-\alpha}<\infty$$

for $0\leqq\alpha<1$ *or if*

$$\int_0^\pi \Phi(t)t^{-2}dt<\infty \quad \textit{and} \quad \sum_{n=1}^\infty m_nP_n^{-1}<\infty,$$

then $\sum m_nA_n(x)\in|N,p_n|$.

3. We shall consider the case that φ is of bounded variation. In this direction we know the following theorem due to R. Mohanty [3]:

Theorem C. *If*

$$\int_0^\pi t^{-\alpha}|d\varphi(t)|<\infty \qquad \textit{for an } \alpha, 0<\alpha<1,$$

then $\sum n^\alpha A_n(x)\in|C,\beta|$ *for every* $\beta>\alpha$.

We generalize this theorem in the following form.

Theorem 5. *Suppose that* (p_n) *and* (m_n) *are sequences satisfying the following conditions*: (i) $p_n\downarrow$ *as* $n\to\infty$, (ii) $m_n\uparrow$ *and* $m_n/n\downarrow$ *as* $n\to\infty$, *and* (iii) $\sum_{k=n}^\infty \dfrac{m_k}{kP_k}\leqq A\dfrac{m_n}{P_n}$. *If* $\int_0^\pi m(1/t)|d\varphi(t)|<\infty$, *then* $\sum m_nA_n(x)\in|N,p_n|$.

We shall prove here only this theorem and the others will be proved in another paper.

4. Proof of Theorem 5. We can suppose that $A_0(x)=0$. By the definition

$$A_j(x)=\frac{1}{\pi}\int_0^\pi \varphi(t)\cos jt\,dt=-\frac{1}{j\pi}\int_0^\pi \sin jt\,d\varphi(t)$$

and then

$$t_n-t_{n-1}=\frac{1}{P_nP_{n-1}}\sum_{j=1}^{n}(P_np_{n-j}-P_{n-j}p_n)m_jA_j(x),$$

$$\sum_{n=1}^{\infty}|t_n-t_{n-1}|\leqq\int_0^{\pi}|d\varphi(t)|\left(\sum_{n=1}^{\infty}\left|\sum_{j=1}^{n}\frac{P_np_{n-j}-P_{n-j}p_n}{P_nP_{n-1}}\frac{m_j}{j}\sin jt\right|\right).$$

It is sufficient to prove that the integrand is less than $A\ m(1/n)$ on $(0,\pi)$. Putting $s=[1/t]$,

$$\sum_{n=1}^{\infty}\left|\sum_{j=1}^{n}\frac{P_np_{n-j}-P_{n-j}p_n}{P_nP_{n-1}}\frac{m_j}{j}\sin jt\right|=\sum_{n=1}^{s}+\sum_{n=s+1}^{\infty}=U+V.$$

Since $P_{n-j}/P_n\uparrow 1$ as $n\to\infty$ for each j,

$$U\leqq t\sum_{n=1}^{s}\sum_{j=1}^{n}m_j\frac{P_np_{n-j}-P_{n-j}p_n}{P_nP_{n-1}}=t\sum_{j=1}^{s}m_j\sum_{n=j}^{s}\left(\frac{P_{n-j}}{P_n}-\frac{P_{n-j-1}}{P_{n-1}}\right)$$

$$=t\sum_{j=1}^{s}m_j\frac{P_{s-j}}{P_s}\leqq A\ m_s.$$

Now

$$V=\sum_{n=s+1}^{\infty}\left|\sum_{j=1}^{n}\frac{P_np_{n-j}-P_{n-j}p_n}{P_nP_{n-1}}\frac{m_j}{j}\sin jt\right|\leqq\sum_{n=s+1}^{\infty}\left|\sum_{j=1}^{s}\right|+\sum_{n=s+1}^{\infty}\left|\sum_{j=s+1}^{n}\right|$$

$$=W+X$$

where

$$W\leqq t\sum_{n=s+1}^{\infty}\sum_{j=1}^{s}m_j\left(\frac{P_{n-j}}{P_n}-\frac{P_{n-j-1}}{P_{n-1}}\right)=t\sum_{j=1}^{s}m_j\left(1-\frac{P_{s-j}}{P_s}\right)\leqq A\ m_s$$

and

$$X\leqq\sum_{n=s+1}^{2(s+1)-1}\left|\sum_{j=s+1}^{[n/2]}\right|+\sum_{n=2(s+1)}^{\infty}\left|\sum_{j=s+1}^{n/2}\right|+\sum_{n=2(s+1)}^{\infty}\left|\sum_{j=[n/2]+1}^{n}\right|$$

$$=X'+Y+Z.$$

$X'\leqq A\ m_s$ similarly as above. Writing $[n/2]=N$,

$$Y=\sum_{n=2(s+1)}^{\infty}\left|\sum_{j=s+1}^{N}\frac{P_np_{n-j}-P_{n-j}p_n}{P_nP_{n-1}}\frac{m_j}{j}\frac{\cos(j-1/2)t\text{-}\cos(j+1/2)t}{2\sin t/2}\right|$$

$$\leqq\sum_{n=2(s+1)}^{\infty}\left(\left|\frac{P_np_{n-N}-P_{n-N}p_n}{P_nP_{n-1}}\frac{m_N}{N}\frac{\cos(N+1/2)t}{2\sin t/2}\right|\right.$$

$$+\left|\sum_{j=s+1}^{N-1}\Delta\left(\frac{m_j}{j}\frac{P_np_{n-j}-P_{n-j}p_n}{P_nP_{n-1}}\right)\frac{\cos(j+1/2)t}{2\sin t/2}\right|$$

$$\left.+\left|\frac{P_np_{n-s-1}-P_{n-s-1}p_n}{P_nP_{n-1}}\frac{m_{s+1}}{s+1}\frac{\cos(s+1/2)t}{2\sin t/2}\right|\right)$$

$$=Y_1+Y_2+Y_3,$$

where

$$Y_1\leqq\frac{A}{t}\sum_{n=2(s+1)}^{\infty}\frac{m_n}{n}\left(\frac{P_{n-N}}{P_n}-\frac{P_{n-N-1}}{P_{n-1}}\right)\leqq\frac{A}{t}\frac{m_s}{s}+\frac{A}{t}\sum_{n=2(s+1)}^{\infty}\Delta\left(\frac{m_n}{n}\right)\leqq A\ m_s.$$

$$Y_2\leqq\frac{A}{t}\sum_{n=2(s+1)}^{\infty}\sum_{j=s+1}^{N-1}\frac{m_j}{j}\left(\frac{p_{n-j-1}}{P_{n-1}}-\frac{p_{n-j}}{P_n}\right)+\Delta\left(\frac{m_j}{j}\right)\left(\frac{P_{n-j-1}}{P_n}-\frac{P_{n-j-2}}{P_{n-1}}\right))$$

$$\leqq\frac{A}{t}\sum_{j=s+1}^{\infty}\sum_{n=2j}^{\infty}\leqq\frac{A}{t}\sum_{j=s+1}^{\infty}\left(\frac{m_j}{j}\frac{p_{j-1}}{P_{2j-1}}+\Delta\left(\frac{m_j}{j}\right)\right)\leqq A\ m_s$$

and similarly $Y_3 \leqq A\, m_s$. Finally we shall estimate Z using the following lemma due to E. Hille and J. D. Tamarkin [4]:

Lemma. *If the sequence q_n is positive and non-increasing, then*

$$\left|\sum_{j=1}^{n} q_j \sin jt\right| \leqq A\, Q(1/t) \quad \textit{and} \quad A\, q_1/t$$

for any n and $t \in (0, \pi)$, where $Q(r) = \sum_{j<r} q_j$ for $r > 1$.

Then we get

$$Z = \sum_{n=2(s+1)}^{\infty} \left| \frac{1}{P_{n-1}} \sum_{j=N+1}^{n} \frac{m_j}{j} p_{n-j} \sin jt - \frac{p_n}{P_n P_{n-1}} \sum_{j=N+1}^{n} \frac{m_j}{j} P_{n-j} \sin jt \right|$$

$$\leqq A\, P_s \sum_{n=2(s+1)}^{\infty} \frac{m_n}{n\, P_{n-1}} + \frac{A}{t} \sum_{n=2(s+1)}^{\infty} \frac{m_n\, p_n}{n\, P_n} \leqq A\, m_s.$$

Thus we have proved the theorem.

References

[1] S. V. Kolhekar: On absolute summability factors for Fourier series. Annales Polonici Mat., **18**, 107–113 (1966).

[2] L. Leindler: On the absolute summability factors of Fourier series. Acta Sci. Mat., **28**, 323–336 (1967).

[3] R. Mohanty: The absolute Cesàro summability of some series associated with a Fourier series and its allied series. J. London Math. Soc., **25**, 63–67 (1950).

[4] E. Hille and J. D. Tamarkin: On the summability of Fourier series. Trans. Am. Math. Soc., **34**, 757–783 (1932).