149. *A Space of Sequences given by Pairs of Unitary Operators*

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1. Introduction. In a recent note [5] on affine transformations with dense orbits R. Sato makes the following statement (Lemma 1). *Let H be a complex (separable) Hilbert space, and let A be a bounded operator and U_1 and U_2 unitary operators on H. Given $\xi, \eta \in H$ there is a complex, regular Borel measure μ on the two-dimensional torus T^2 whose Fourier-Stieltjes transform is given by*

(1) $$\hat{\mu}(m,n) = \langle AU_1^m\xi, U_2^n\eta\rangle, \qquad -\infty < m, n < \infty.$$

The purpose of this paper is to point out some counterexamples to this proposition and to examine more carefully the class of sequences of the type appearing on the right-hand side of (1).

We refer the reader to [4] for some background and related results on affine transformations. Here let us just recall that doubly-indexed sequences of the type indicated in (1) arise in the study of affine transformations on locally compact groups as follows. Let G be a locally compact group and $\tau(x)$ a *bi*-continuous, Haar-measure-preserving automorphism of G. Let $a \in G$, and consider the affine transformation $T(x) = a\tau(x), x \in G$. Denote the left regular representation of G on $L^2(G)$ by V, and let U_1 and U_2 be the unitary operators on $L^2(G)$ given by composition with $T(x)$ and $\tau(x)$, respectively.

Lemma [5]. $U_2^{-1}VU_1 = V\circ T$. *Thus for $f, g \in L^2(G)$ we have*

$$\langle V\circ T^n(x)f, g\rangle = \langle V(x)U_1^n f, U_2^n g\rangle, \qquad -\infty < n < \infty,\ x \in G.$$

The fact that a measure μ satisfying (1) need not exist for all choices of A, U_1 and U_2 is immediate from the following

Proposition. *Let $(a_n)_{n=-\infty}^{\infty}$ be any bounded sequence of complex numbers. There exist a bounded operator A and a unitary operator U on the Hilbert space H and $\xi, \eta \in H$ such that*

$$a_n = \langle AU^n\xi, U^n\eta\rangle, \qquad -\infty < n < \infty.$$

Proof. Let H denote the bilateral sequence space $l_2(-\infty, \infty)$ with standard basis $\{\cdots, e_{-1}, e_0, e_1, \cdots\}$. Let U be the bilateral shift operator: $Ue_n = e_{n+1}$, $-\infty < n < \infty$, and let A be the bounded operator on H given by coordinatewise multiplication with the given sequence $(a_n)_{n=-\infty}^{\infty}$. Then

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$$\langle AU^n e_0, U^n e_0\rangle = \langle a_n e_n, e_n\rangle = a_n \qquad -\infty < n < \infty.$$

2. Sequences given by unitary operators. Let $\mathcal{S}$ denote the space of all sequences $(\langle AU_1^m\xi, U_2^n\eta\rangle)_{m,n=-\infty}^{\infty}$ for A a bounded operator and U_1 and U_2 unitary operators on some Hilbert space H and $\xi, \eta \in H$. The following lemma shows that the operator A may be suppressed in studying $\mathcal{S}$.

Lemma. *Let $(\alpha_{mn})_{m,n=-\infty}^{\infty} \in \mathcal{S}$. There exist unitary operators V_1 and V_2 on a Hilbert space K and $\xi', \eta' \in K$ such that*

$$(2) \qquad \alpha_{mn} = \langle V_1^m\xi', V_2^n\eta'\rangle, \qquad -\infty < m, n < \infty.$$

Proof. Let the α_{mn} be given by operators A, U_1, U_2 as above on some Hilbert space H and $\xi, \eta \in H$, and let $c = \|A\| \neq 0$. Let W be a unitary dilation of $c^{-1}A$ on the space K containing H [3]. Set $\xi' = c\xi$, $\eta' = w^*\eta$, and $V_2 = W^*\tilde{U}_2W$, where $\tilde{U}_2$ is any unitary extension of U_2 to K, and let V_1 be any extension of U_1 to a unitary operator on K. Then

$$\begin{aligned}\alpha_{mn} &= \langle cWU_1^m\xi, U_2^n\eta\rangle = \langle cU_1^m\xi, W^*U_2^n\eta\rangle \\ &= \langle cU_1^m\xi, (W^*U_2^nW)(W^*\eta)\rangle \\ &= \langle V_1^m\xi', V_2^n\eta'\rangle, \qquad -\infty < m, n < \infty.\end{aligned}$$

Theorem. *$\mathcal{S}$ is a conjugate-closed algebra of bounded, doubly-indexed sequences containing all the sequences of Fourier-Stieltjes coefficients of complex Borel measures on the torus.*

Proof. Let $(\alpha_{mn})_{m,n=-\infty}^{\infty}, (\beta_{mn})_{m,n=-\infty}^{\infty} \in \mathcal{S}$ be given as in (2) in terms of U_1, U_2, ξ, η and V_1, V_2, ξ', η', on some Hilbert spaces H and K, respectively. On the Hilbert space $H \oplus K U_1 \oplus V_1$ and $U_2 \oplus V_2$ are unitary, and we have

$$\begin{aligned}\alpha_{mn} + \beta_{mn} &= \langle U_1^m\xi, U_2^n\eta\rangle_H + \langle V_1^m\xi', V_2^n\eta'\rangle_K \\ &= \langle (U_1^m\xi, V_1^m\xi'), (U_2^n\eta, V_2^n\eta')\rangle_{H\oplus K} \\ &= \langle (U_1\oplus V_1)^m(\xi, \xi'), (U_2\oplus V_2)^n(\eta, \eta')\rangle_{H\oplus K}, \qquad -\infty < m, n < \infty.\end{aligned}$$

Thus $(\alpha_{mn} + \beta_{mn})_{m,n=-\infty}^{\infty} \in \mathcal{S}$. To show $\mathcal{S}$ is closed under multiplication consider the Hilbert-space tensor product $H \hat{\otimes} K$ [6]. Again, $U_1 \otimes V_1$ and $U_2 \otimes V_2$ are unitary, and

$$\begin{aligned}\alpha_{mn}\beta_{mn} &= \langle U_1^m\xi, U_2^n\eta\rangle_H \langle V_1^m\xi', V_2^n\eta'\rangle_K \\ &= \langle U_1^m\xi \otimes V_1^m\xi', U_2^n\eta \otimes V_2^n\eta'\rangle_{H\hat{\otimes}K} \\ &= \langle (U_1\otimes V_1)^m(\xi\otimes\xi'), (U_2\otimes V_2)^n(\eta\otimes\eta')\rangle_{H\hat{\otimes}K}, \qquad -\infty < m, n < \infty.\end{aligned}$$

Now let μ be a complex Borel measure on the torus, and let f be a function of unit modulus such that $d\mu = fd|\mu|$. On $L^2(|\mu|)$ let $U_1 g = \bar{Z}_1 g$ and $U_2 g = Z_2 g$. Then

$$\begin{aligned}\hat{\mu}(m, n) &= \int \bar{Z}_1^m \bar{Z}_2^n d\mu(Z_1, Z_2) = \int \bar{Z}_1^m \bar{Z}_2^n f d|\mu| \\ &= \langle U_1^m f, U_2^n 1\rangle, \qquad -\infty < m, n < \infty.\end{aligned}$$

Consider the tensor algebra $V = C(T)\hat{\otimes}_\gamma C(T)$, where γ denotes the greatest cross norm [6]. We now show that $\mathcal{S}$ can be embedded in the dual space V^* of V. Given $(\alpha_{mn})_{m,n=-\infty}^{\infty} \in \mathcal{S}$, represented as in (2), and trigonometric polynomials

$$p(\theta)=\sum_{m=-M}^{M} a_m e^{im\theta} \quad \text{and} \quad q(\theta)=\sum_{n=-N}^{N} b_n e^{in\theta},$$

set

$$(3) \qquad F(p\otimes q)=\sum_{m=-M}^{M}\sum_{n=-N}^{N} a_m b_n \alpha_{-m,n}.$$

Then

$$(4) \qquad \begin{aligned} |F(p\otimes q)| &= \Big|\sum_{m,n} a_m b_n \langle U_1^{-m}\xi, U_2^n\eta\rangle\Big| \\ &= \Big|\Big\langle \sum_{m=-M}^{M} a_m U_1^{-m}\xi, \sum_{n=-N}^{N} \bar{b}_n U_2^n \eta\Big\rangle\Big| \\ &= |p(U_1^{-1})\xi, q^*(U_2)\eta| \le \|p(U_1^{-1})\xi\|\,\|q^*(U_2)\eta\| \\ &\le \|p\|\|q\|\|\xi\|\|\eta\|. \end{aligned}$$

(Here $q^*(\theta)=\overline{q(-\theta)}$.) Thus F can be extended uniquely to an element of V^* with norm at most $\|\xi\|\|\eta\|$. The following is now clear.

Theorem. *The mapping* $(\alpha_{mn})_{m,n=-\infty}^{\infty}\to F$, *as in* (3), *is a linear embedding of* $\mathcal{S}$ *into* V^*. *The Fourier-Stieltjes sequences are precisely those which correspond, under this embedding, to the elements of* V^* *which are continuous in the uniform norm* (*least cross norm*).

Remark. It would be interesting to determine whether or not every functional in V^* can be obtained in this way from $\mathcal{S}$. If this were the case, then (3) and (4) would represent an analog for $\mathcal{S}$ of a well-known criterion which characterizes Fourier-Stieltjes sequences.

Our next theorem shows that if a sequence in $\mathcal{S}$ is the Fourier-Stieltjes transform of some measure on T^2 this measure can be described in terms of the spectral measures of the unitary operators involved. Before stating this result, let us examine V^* a bit more carefully. Elements of V^* will be called *bimeasures*, in accordance with [1], [2]. Briefly, for every $F\in V^*$ there is a unique function $k(\theta,\varphi)$ on T^2 (which we also call a bimeasure) which vanishes for $\theta=0$ or $\varphi=0$, is left continuous in each of the variables, has finite *Fréchet variation*, and satisfies

$$(5) \qquad F(f\otimes g)=\int_T f(\theta)d_\theta\int_T g(\varphi)d_\varphi k(\theta,\varphi), \qquad f,g\in C(T).$$

The Fréchet variation $\Phi(k)$ of k is defined as follows. Given partitions

$$0=s_1<s_2<\cdots<s_p=2\pi \quad \text{and} \quad 0=t_1<t_2<\cdots<t_q=2\pi$$

of T we set, for $1\le i\le p$ and $1\le j\le q$,

$$\Delta_{ij}k=k(s_i,t_j)-k(s_{i-1},t_j)-k(s_i,t_{j-1})+k(s_{i-1},t_{j-1}).$$

Then

$$\Phi(k)=\sup\Big|\sum_{i=1}^{p}\sum_{j=1}^{q}\alpha_i\beta_j\Delta_{ij}k\Big|,$$

the supremum being taken over all pairs of partitions and corresponding sequences $\alpha_1,\cdots,\alpha_p$ and $\beta_1,\cdots,\beta_q$ of complex numbers of modulus at most one.

Theorem. *Let* U_1, U_2, ξ, η *be as above on some Hilbert space,* *and let E_i denote the spectral measure for U_i on $[0,2\pi)$, $i=1,2$. For X and Y Borel sets, let*

$$\nu(X\times Y)=\langle E_1(X)\xi, E_2(Y)\eta\rangle.$$

Then there exists a measure μ on T^2 satisfying (1) (with A having been absorbed as in our first lemma) if and only if ν can be extended to a measure on T^2.

Proof. If ν can be extended to a measure, then a computation shows that $\hat{\nu}(m,n)=\alpha_{-m,n}$. Conversely, suppose we can find a measure μ such that $\hat{\mu}(m,n)=\alpha_{-m,n}$, $-\infty<m,n<\infty$. It is easy to see that the bimeasure corresponding to μ is the function

$$k(\theta,\varphi)=\mu([0,\theta)\times[0,\varphi)).$$

On the other hand, let

$$h(\theta,\varphi)=\langle E_1([0,\theta))\xi, E_2([0,\varphi))\eta\rangle.$$

Then h is left continuous in both θ and φ, and given a pair of partitions as above we have

$$\Delta_{ij}h=\langle E_1([s_{i-1},s_i))\xi, E_2([t_{j-1},t_j))\eta\rangle.$$

Thus for $|\alpha_i|\le 1, |\beta_i|\le 1$,

$$\begin{aligned}\left|\sum_{i=1}^{p}\sum_{j=1}^{q}\alpha_i\beta_j\Delta_{ij}h\right|&=\left|\left\langle \sum_{i=1}^{p}\alpha_iE_1([s_{i-1},s_i))\xi, \sum_{j=1}^{q}\bar{\beta}_jE_2([t_{j-2},t_j))\eta\right\rangle\right|\\&\le\left\|\sum_{i=1}^{p}\alpha_iE_1([s_{i-1},s_i))\xi\right\|\left\|\sum_{j=1}^{q}\bar{\beta}_jE_2([t_{j-1},t_j))\eta\right\|\\&\le\|\xi\|\|\eta\|.\end{aligned}$$

Hence $\Phi(h)\le\|\xi\|\|\eta\|$, so h defines a bimeasure, namely

$$\begin{aligned}&\int_0^{2\pi}f(\theta)d_\theta\int_0^{2\pi}g(\varphi)d_\varphi h(\theta,\varphi)\\&\quad=\int_0^{2\pi}f(\theta)d_\theta\left\langle E_1([0,\theta))\xi, \int_0^{2\pi}\overline{g(\varphi)}dE_2(\varphi)\eta\right\rangle\\&\quad=\left\langle\int_0^{2\pi}f(\theta)dE_1(\theta)\xi, \int_0^{2\pi}\overline{g(\varphi)}dE_2(\varphi)\eta\right\rangle, \qquad f,g\in C(T).\end{aligned}$$

It is clear from the Spectral Theorem that h corresponds as in the previous theorem to $\hat{\mu}$, as, of course, does k also. By the uniqueness of the representation (5), we must have $h=k$, which means ν can be extended to a measure on T^2.

Remark. Note that by the uniqueness of the representation (5) the function h above depends only on the sequence in $\mathcal{S}$ and not on the choice of the unitary operators defining it.

Corollary. *Let $(\alpha_{mn})_{m,n=-\infty}^{\infty}$ be given as in (2). Then it is a Fourier-Stieltjes sequence if and only if the following condition holds: For any pair of partitions $\{X_i\}_{i=1}^{\infty}$ and $\{Y_j\}_{j=1}^{\infty}$ of $[0,2\pi)$,*

$$\sum_{i,j=1}^{\infty}|\langle E_1(X_i)\xi, E_2(Y_j)\eta\rangle|<\infty. \tag{6}$$

Proof. If $(\alpha_{mn})_{m,n=-\infty}^{\infty}$ is a Fourier-Stieltjes sequence, then consideration of the measure $|\nu|$, whose existence is implied by our theorem, makes it clear that (6) must hold for any pair of partitions.

Conversely, note that for any Borel sets X and Y, $\nu(\cdot, Y)$ and $\nu(X, \cdot)$ are measures. Using this fact and (6) it can be shown that ν is countably additive on the algebra of finite unions of measurable rectangles.

Example. Suppose that for $i=1,2$ H has a basis $\{e_j^i\}_{j=1}^{\infty}$ of eigenvectors of U_i corresponding to distinct eigenvalues. Let $\xi, \eta \in H$, and write $\xi=\sum_{i=1}^{\infty} x_i e_i^1$ and $\eta=\sum_{j=1}^{\infty} y_j e_j^2$. Then $(\alpha_{mn})_{m,n=-\infty}^{\infty}$ *(as in* (2)) *is a Fourier-Stieltjes sequence if and only if*

$$\sum_{i,j=1}^{\infty} |x_i|\,|y_i|\,|\langle e_i^1, e_j^2 \rangle| < \infty.$$

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