148. *Ergodic Automorphisms and Affine Transformations*

By M. RAJAGOPALAN[1] and Bert M. SCHREIBER[2),3]

(Comm. by Kinjirô KUNUGI, M. J. A., Sept. 12, 1970)

By modifying an argument which originated with the authors R. Sato [6] has attempted to prove that *if T is a continuous affine transformation of a locally compact group G such that the orbit $\{T^n(x): -\infty<n<\infty\}$ of some element x is dense in G, then G is compact.* That paper was suggested by a series of papers aimed at answering the following question first raised by Halmos [2, p. 29]. Can an automorphism of a locally compact but non-compact group be an ergodic measure-preserving transformation?

As pointed out in [4], the stated key lemma (Lemma 1) of the argument in [6] does not in fact hold. The question of Halmos, as well as the results stated in [6] and [7, I, Theorem 3], thus remain open questions. The purpose of this note is to announce, in § 1, some results bearing on the question of Halmos and on the analogous one for affine transformations. The proofs will appear in [5]. In § 2 we answer in the affirmative a question raised by Sato in [7, II].

1. Groups with ergodic transformations. Although the theorems below are stated in the context of ergodic, measure-preserving transformations, all the results remain valid if ergodicity is replaced by the assumption that the orbit of some element is dense. G will denote a locally compact group and G_0 its identity component. By an affine transformation on G we mean a mapping of the form $T(x)=a\tau(x)$, where τ is a bi-continuous automorphism of G and $a\in G$.

Theorem. *Let G have an ergodic automorphism. If G/G_0 is compact, then G must be compact. Thus if there exists a noncompact group with an ergodic automorphism then there exists a noncompact totally disconnected one.*

Theorem. *Let G be totally disconnected, and assume it has an ergodic automorphism. If G also satisfies one of the following conditions, then it must be compact:*

(i) *Every compact subset of G is contained in a compact subgroup.*

(ii) *G has a compact, open, normal subgroup.*

---

1) Department of Mathematics, Madurai University, Madurai-2, Tamilnad, India.

2) Department of Mathematics, Wayne State University, Detroit, Michigan.

3) Research supported by National Science Foundation Grant No. GP-13741.

(iii) *G is nilpotent.*

(iv) *G is maximally almost periodic.*

Theorem. *Let k be a nondiscrete, locally compact, totally disconnected field, and let $G(k)$ denote the rational elements over k of a connected (in the Zariski topology), semi-simple, linear algebraic group G defined over k [1]. Then $G(K)$ cannot have an ergodic inner automorphism. If* $\mathrm{char}(k)=0$ *then $G(k)$ has no ergodic automorphisms.*

When automorphisms are replaced by arbitrary affine transformations, our results are analogous, though even less complete:

Theorem. *If G has an ergodic affine transformation, then it is compactly generated. If G has a nontrivial, compact, open, normal subgroup, then G must be compact.*

Corollary. *If G is discrete and has an affine transformation with only one orbit, then G is finitely generated.*

Corollary. *If G is maximally almost periodic and has an ergodic affine transformation, then G is compact or discrete.*

Theorem. *If G is nilpotent (in particular, abelian) and has an ergodic affine transformation, then $G=Z$ or G is compact.*

Remark. D. Jonah and the last author have recently shown that the only infinite, nonabelian group having an affine transformation with only one orbit is the dihedral group $D_\infty = Z \circledS Z_2$.

2. Ergodicity and dense orbits on compact groups. Let G be a compact group and $T(x)=a\tau(x)$ an affine transformation on G. It is well known that if G is metrizable and T is ergodic, then some (in fact almost every) element of G has a dense orbit under T. The converse of this fact, namely, that a dense orbit implies ergodicity, was first proved by P. Walters [8] in case G is abelian and connected. His proof is an application of a theorem of A. H. M. Hoare and W. Parry which gives necessary and sufficient conditions for an affine transformation on a compact, abelian, connected group to be ergodic [3]. Another approach to the abelian case was announced by Sato [7, II], who then asked whether a dense orbit implies ergodicity in the nonabelian case. In this section we shall show that it does.

The following lemma is purely algebraic; no topology is involved. The details are easy to compute.

Lemma. *Let T be a one-to-one mapping of a group G onto itself.* Then T is an affine transformation if and only if

$$T(xy^{-1}z)=T(x)(T(y))^{-1}T(z), \qquad x,y,z\in G.$$

Returning to the compact group G, we shall denote by dx a normalized Haar measure on G and by $f*g$ the convolution of $f,g\in L^2(G)$. For $f\in L^2(G)$ we set $\tilde{f}(x)=\overline{f(x^{-1})}$.

Lemma. *If h is a nonconstant function in $L^2(G)$, then $h*\tilde{h}$ is not constant.*

Proof. If $h*\tilde{h}$ is constant, that constant must be

$$\int_G h*\tilde{h}(x)dx=\int_G\int_G h(xy)\overline{h(y)}dydx$$
$$=\int_G h(x)\,dx\int_G \overline{h(y)}dy=\left|\int_G h(x)\,dx\right|^2.$$

But

$$h*\tilde{h}(1)=\int_G |h(y)|^2dy>\left|\int_G h(y)\,dy\right|^2,$$

so $h*\tilde{h}$ is not constant.

Theorem. *Let G be a compact group and $T(x)=a\tau(x)$ an affine transformation. If $\{T^n(x): -\infty<n<\infty\}$ is dense in G for some element x, then T is ergodic.*

Proof. Suppose T is not ergodic. It suffices to construct a nonconstant, continuous, T-invariant function on G. To do so let f be a nonconstant function in $L^2(G)$ with $f\circ T=f$ a.e. Set $g=f*\tilde{f}*f$. Then g is continuous as a convolution of L^2 functions. And since T is measure preserving we have from the first lemma,

$$g(T(x))=\int_G f*\tilde{f}(T(x)y)f(y^{-1})dy$$
$$=\int_G f*\tilde{f}(T(x)y^{-1})f(y)dy$$
$$=\int_G\int_G f(T(x)y^{-1}z)\overline{f(z)}f(y)dzdy$$
$$=\int_G\int_G f(T(x)(T(y))^{-1}T(z))\overline{f(T(z))}f(T(y))dzdy$$
$$=\int_G\int_G f\circ T(xy^{-1}z)\overline{f\circ T(z)}f\circ T(y)dzdy$$
$$=\int_G\int_G f(xy^{-1}z)\overline{f(z)}f(y)dzdy$$
$$=g(x).$$

To complete the proof we must show g is nonconstant. By our second lemma $f*\tilde{f}$ is nonconstant, whence so is

$$g*\tilde{f}=(f*\tilde{f})*(f*\tilde{f})^{\tilde{}}.$$

But if g were constant $g*\tilde{f}$ would also be.

References

[1] A. Borel: Linear Algebraic Groups. Benjamin, New York (1969).

[2] P. R. Halmos: Lectures on Ergodic Theory. Pub. Math. Soc., Japan, Kenkyusha, Tokyo (1956).

[3] A. H. M. Hoare and W. Parry: Semi-groups of affine transformations. Quarterly J. Math., **17** (2), 106–111 (1966).

[4] T. Ito and B. M. Schreiber: A space of sequences given by pairs of unitary

operators. Proc. Japan Acad., **46**, 637–641 (1970).
[5] M. Rajagopalan and B. Schreiber: Ergodic automorphisms and affine transformations of locally compact groups (to appear).
[6] R. Sato: Continuous affine transformations of locally compact totally disconnected groups. Proc. Japan Acad., **46**, 143–146 (1970).
[7] ——: Properties of ergodic affine transformations of locally compact groups. I, II. Proc. Japan Acad., **46**, 231–238 (1970).
[8] P. Walters: Topological conjugacy of affine transformations of tori. Trans. Amer. Math. Soc., **131**, 40–50 (1968).